CPM 2013

Converting SLP to LZ78 in almost Linear Time

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"Recompress" SLP to LZ78

- ✓ The problem we consider is kind of recompression, i.e., convert SLP for string w to LZ78 encoding of w.

- ✓ Example of our motivation:
  - ➢ NCD of two strings s and t w.r.t. LZ78 compressor: Determined by $|\mathrm{LZ78}(s)|$, $|\mathrm{LZ78}(t)|$, and $|\mathrm{LZ78}(st)|$.
  - ➢ See [Bannai et al., SPIRE 2012] for more.

- ✓ Why LZ78?
  - ➢ Widely used (GIF, PDF, TIFF, etc.).
  - ➢ Nice CSP algorithms (cf., [Gawrychowski 2011, 2012]).

Straight Line Program (SLP)

An SLP is a sequence of productions

$$X_1 = expr_1,\ X_2 = expr_2,\ \cdots,\ X_n = expr_n$$

- $expr_i = a \qquad (a \in \Sigma)$
- $expr_i = X_l X_r \quad (l, r < i)$

✓ The size of the SLP is the number n of productions.

✓ An SLP is essentially a CFG deriving a single string.

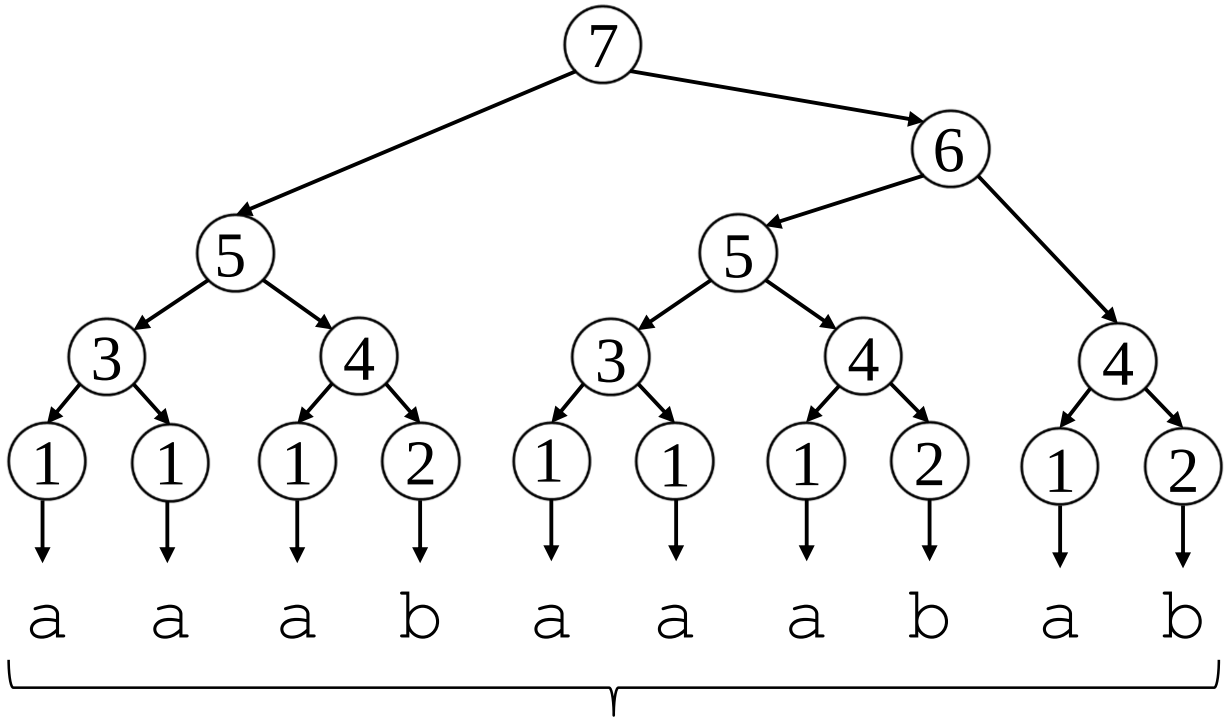
✓ SLPs model outputs of grammar-based compression algorithms (e.g., Re-pair, Sequitur, LCAcomp, etc).

Example of SLP

SLP G

$X_1 = \mathtt{a}$
$X_2 = \mathtt{b}$
$X_3 = X_1 X_1$
$X_4 = X_1 X_2$
$X_5 = X_3 X_4$
$X_6 = X_5 X_4$
$X_7 = X_5 X_6$

derivation tree for G



string w represented by G

DAG Representation of SLP

SLP G

$X_1 = \mathrm{a}$
$X_2 = \mathrm{b}$
$X_3 = X_1 X_1$
$X_4 = X_1 X_2$
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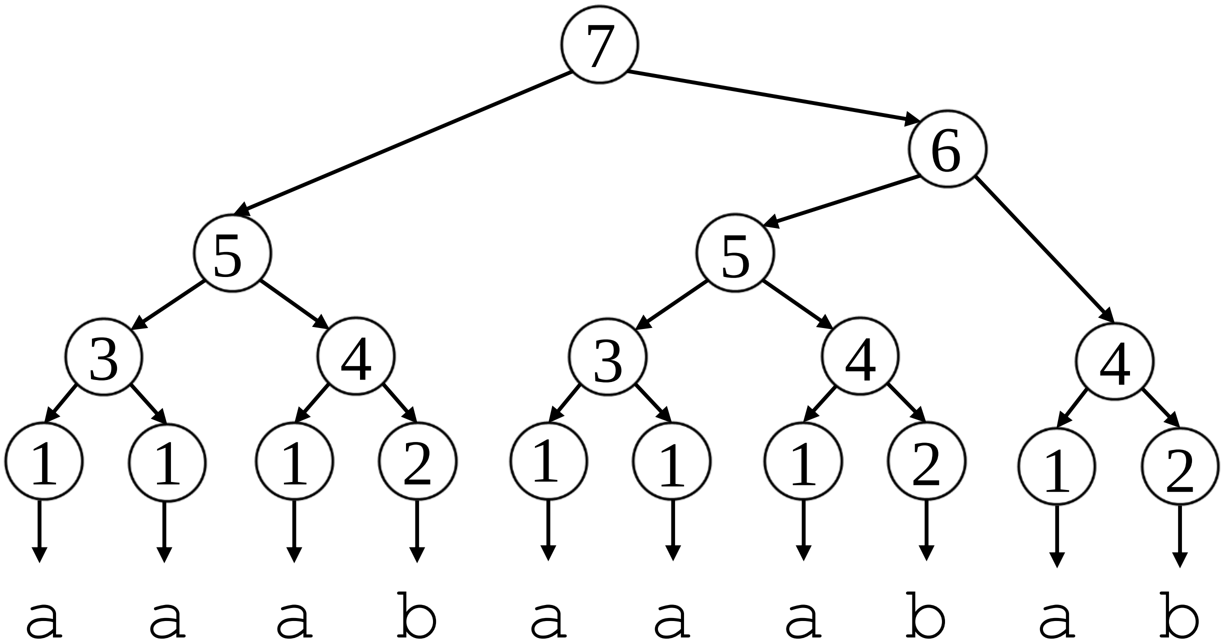

- ✓ DAG is compressed representation of derivation tree.
- ✓ SLP is compressed representation of string.

LZ78 Factorization

The LZ78 factorization of string w is a factorization

$$w = f_1 f_2 \dots f_m$$

where f_j is the longest prefix of $f_j \dots f_m$ such that $f_j = f_k c$ for some $0 \leq k < j$ (let $f_0 = \varepsilon$) and $c \in \Sigma$.

$w =$ a a a b a a a b a b

LZ78 trie of w

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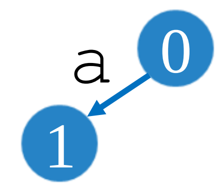

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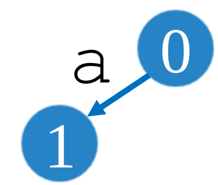
$w =$ a | a a | b a a a b a b

f_1 | f_2 |

LZ78 trie of w



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$w =$ a | a a | b a a a b a b

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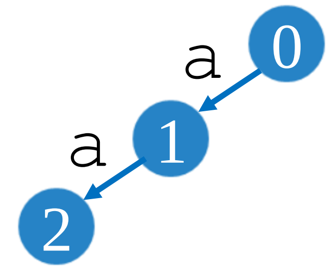
LZ78 trie of w



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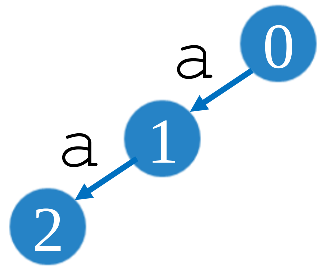
$w =$ a | a a | b | a a a b a b

f_1 | f_2 | f_3 |

LZ78 trie of w



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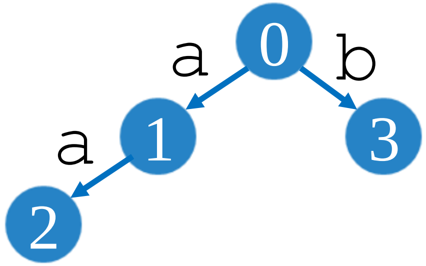
$w =$ a | a a | b | a a a b a b

f_1 | f_2 | f_3 |

LZ78 trie of w



LZ78 Factorization

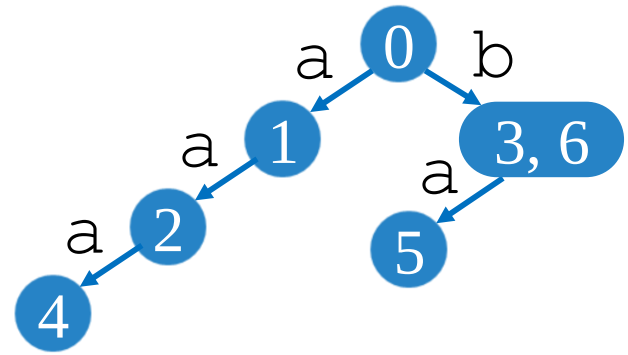
The LZ78 factorization of string w is a factorization

$$w = f_1 f_2 \ldots f_m$$

where f_j is the longest prefix of $f_j \ldots f_m$ such that $f_j = f_k c$ for some $0 \le k < j$ (let $f_0 = \varepsilon$) and $c \in \Sigma$.

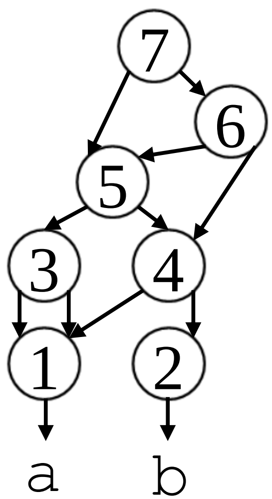
$$w = \underbrace{\mathtt{a}}_{f_1} \mid \underbrace{\mathtt{a\,a}}_{f_2} \mid \underbrace{\mathtt{b}}_{f_3} \mid \underbrace{\mathtt{a\,a\,a}}_{f_4} \mid \underbrace{\mathtt{b\,a}}_{f_5} \mid \underbrace{\mathtt{b}}_{f_6}$$

LZ78 trie of w



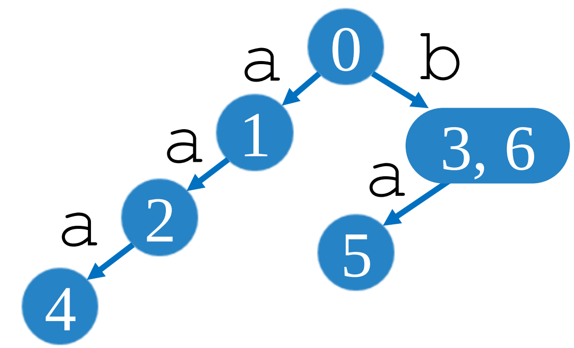
Converting SLP to LZ78 [Cont.]

SLP G of size n

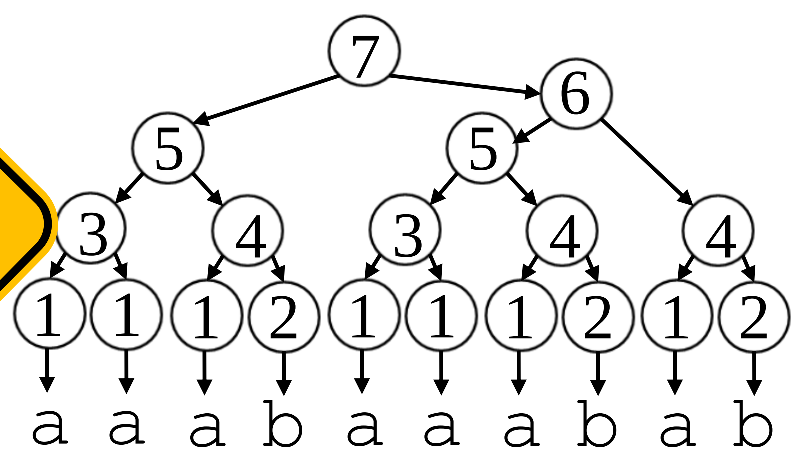


Go this way!

LZ78 trie of size m



derivation tree of G



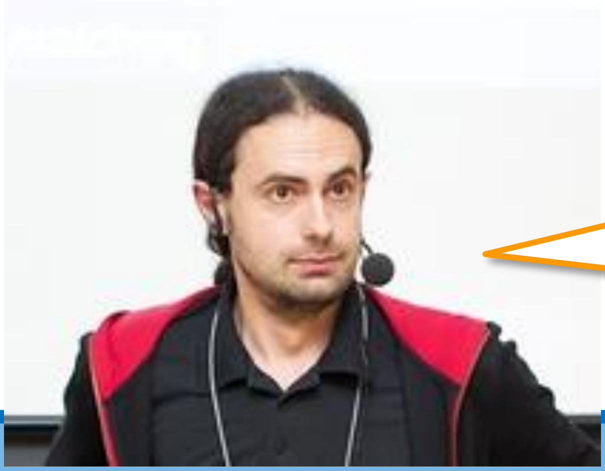
$O(2^n)$ time!!

Previous Result & our Hero Joined

Theorem 1 [Bannai, Inenaga, Takeda, SPIRE 2012]

Given an SLP of size n describing string w, we can compute the LZ78 trie of w of size m in $O(nL + m \log N)$ time.

- N is the length of uncompressed string w
- L is the length of longest LZ78 factor of w



Hmmm, this can be improved!

New Result

Theorem 2 (new!)

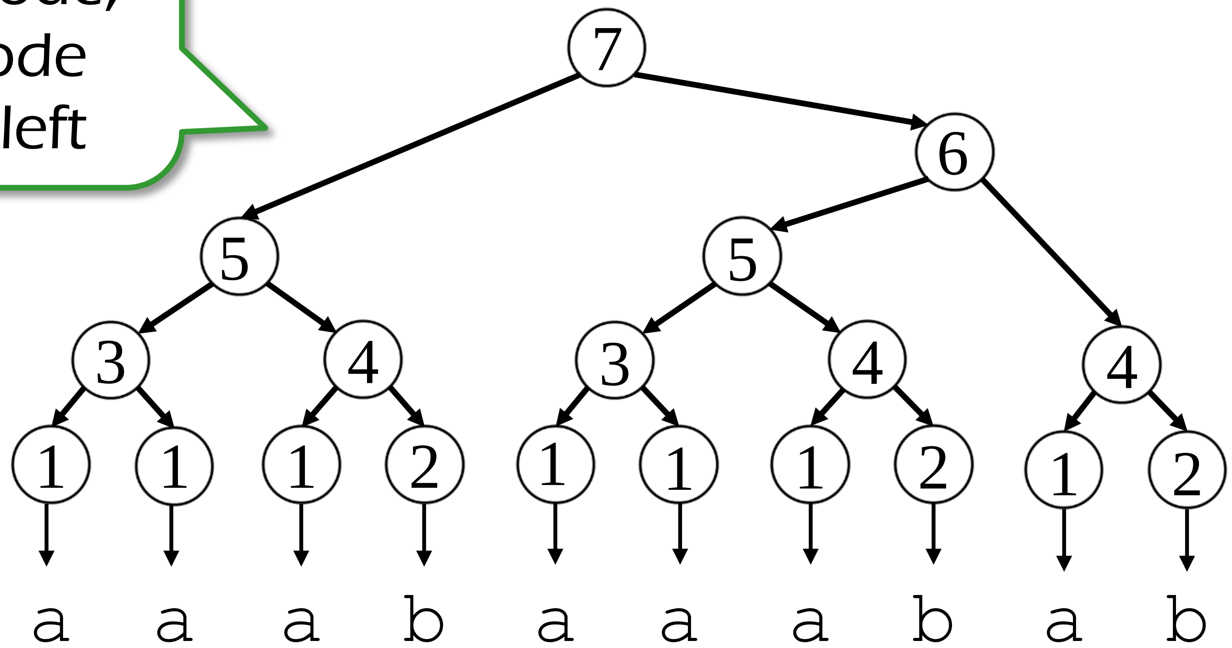
Given an SLP of size n describing string w, we can compute the LZ78 trie of w of size m in $\underline{O(n + m \log m)}$ time.

- ✓ Improved from $O(nL + m \log N)$ to $O(n + m \log m)$
  - $L = O(\sqrt{N})$ is the length of the longest LZ78 factor

G-parsing

Prune the subtree rooted at internal node, if the label of the node also appears to the left

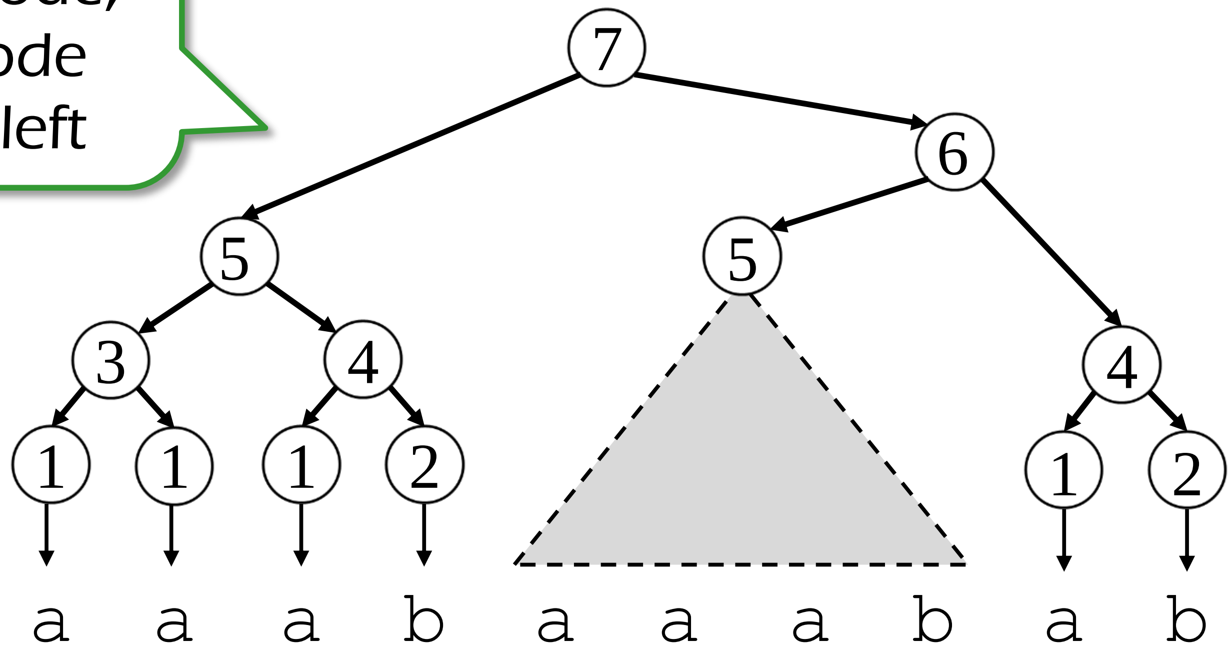
derivation tree for SLP *G*



G-parsing

Prune the subtree rooted at internal node, if the label of the node also appears to the left

derivation tree for SLP G



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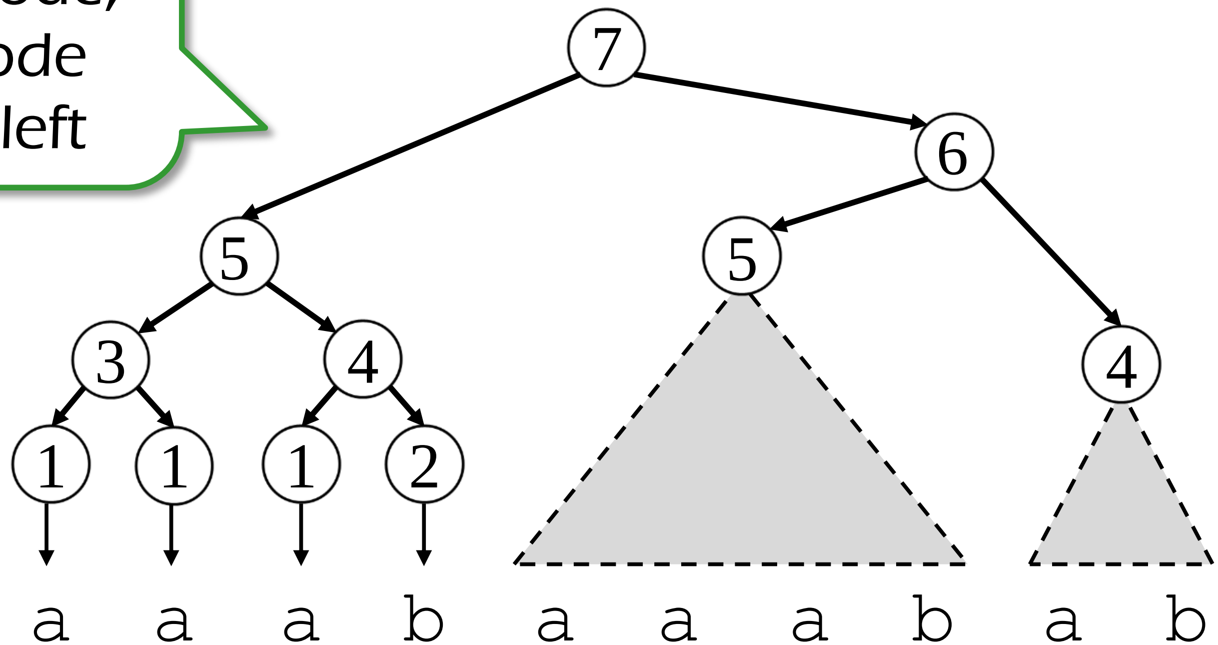

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Prune the subtree rooted at internal node, if the label of the node also appears to the left

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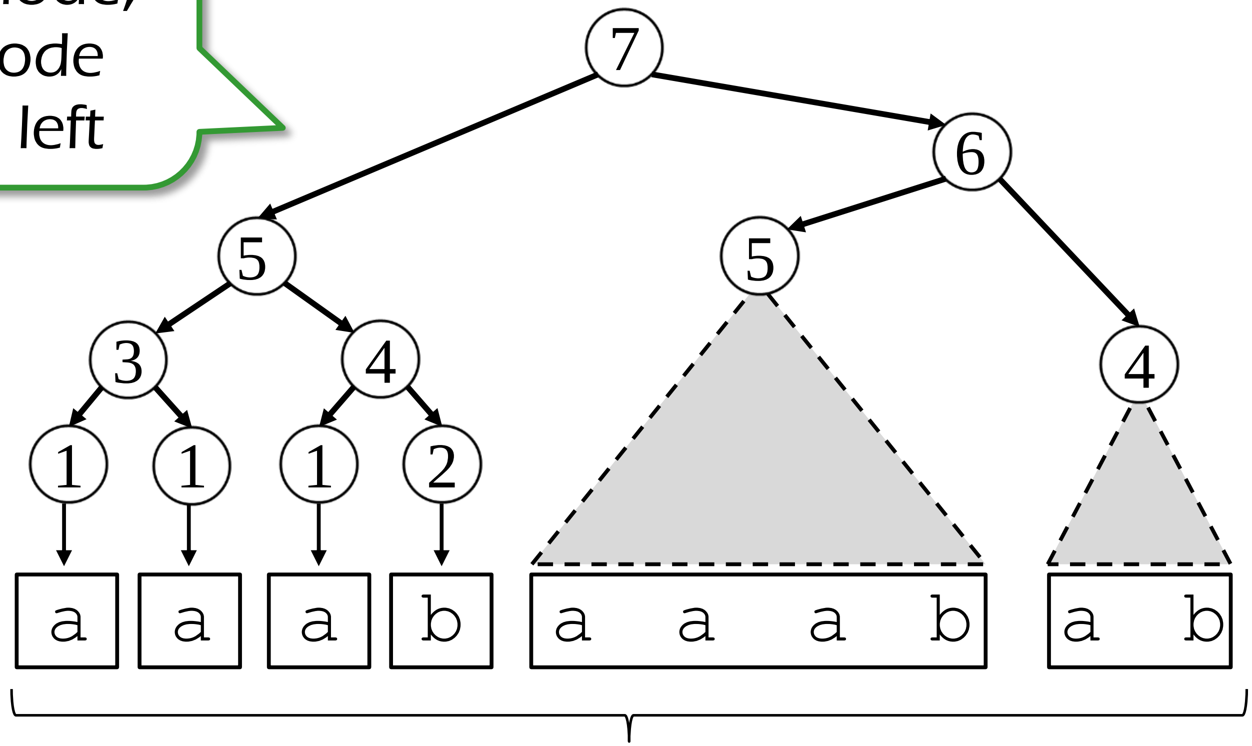


G-parsing of string *w*

G-parsing [Cont.]

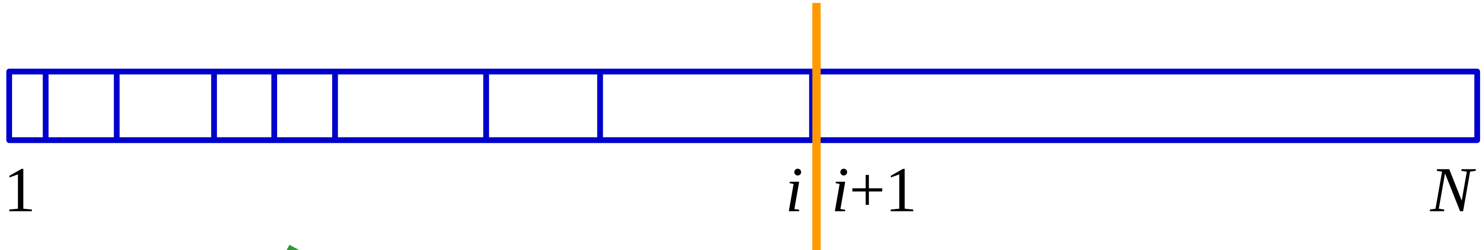
Lemma 1 [Rytter 2003]

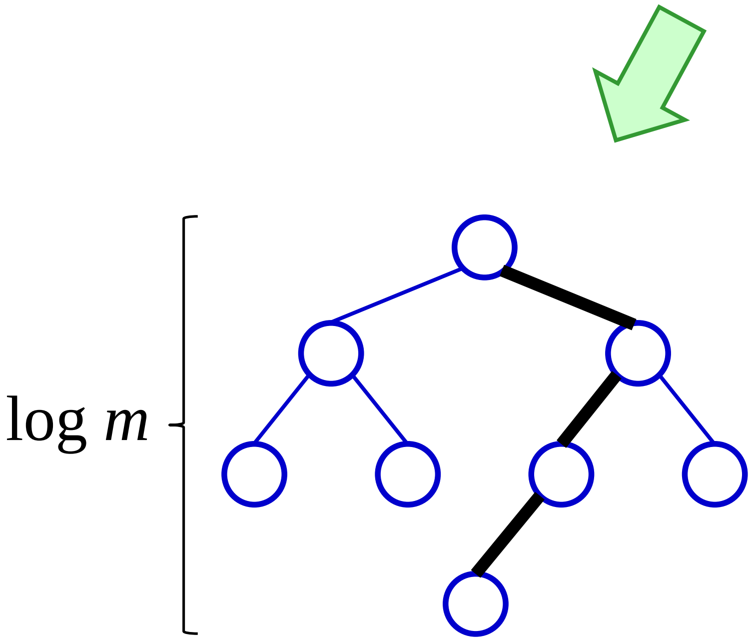
For SLP G of size n, the G-parsing contains $O(n)$ blocks and can be computed in $O(n)$ time.

✓ Each block of G-parsing is represented by its beginning and ending positions, thus taking $O(1)$ space.

LZ78 Factorization on SLP

LZ factors



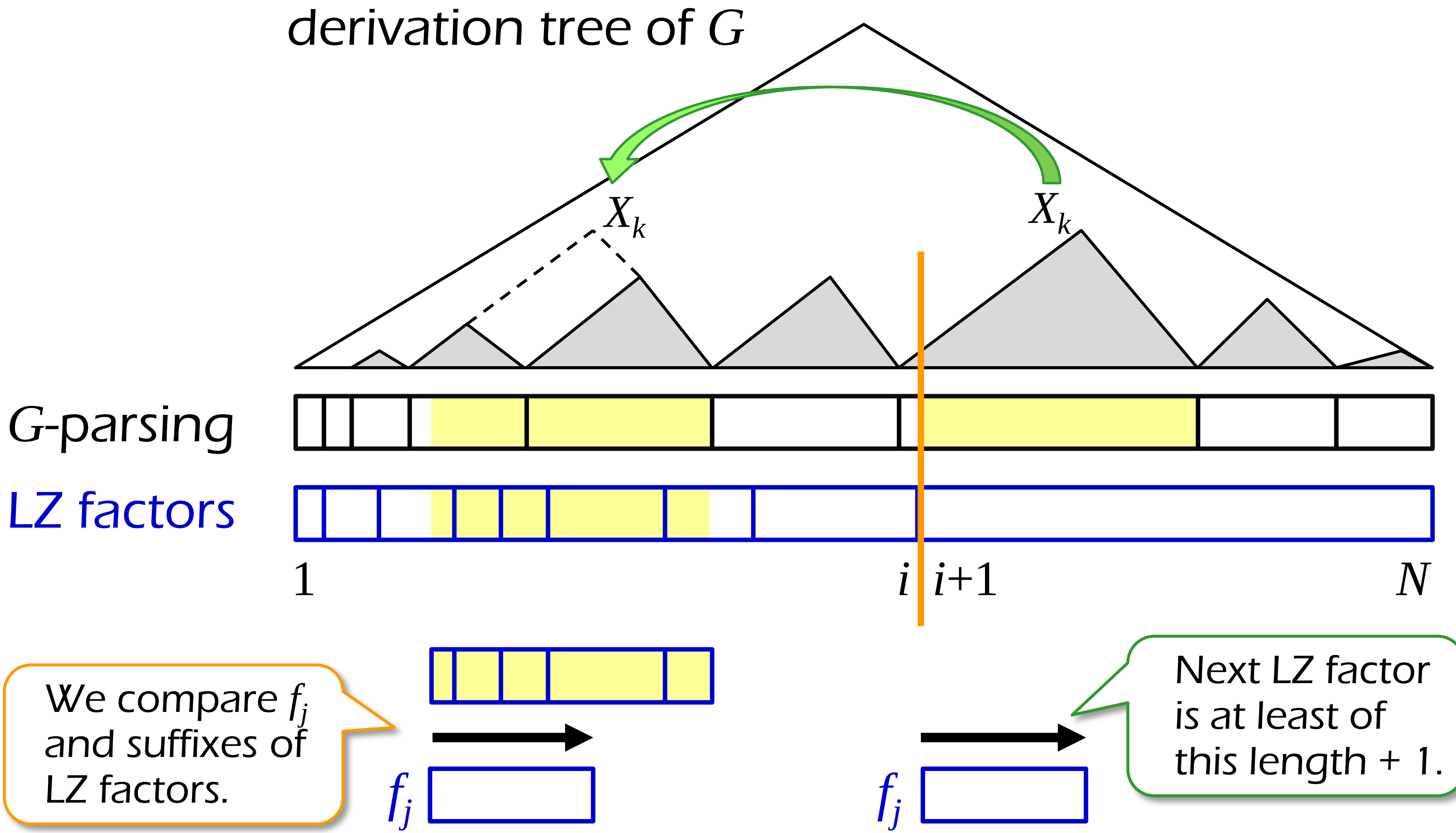


BST for previous LZ factors

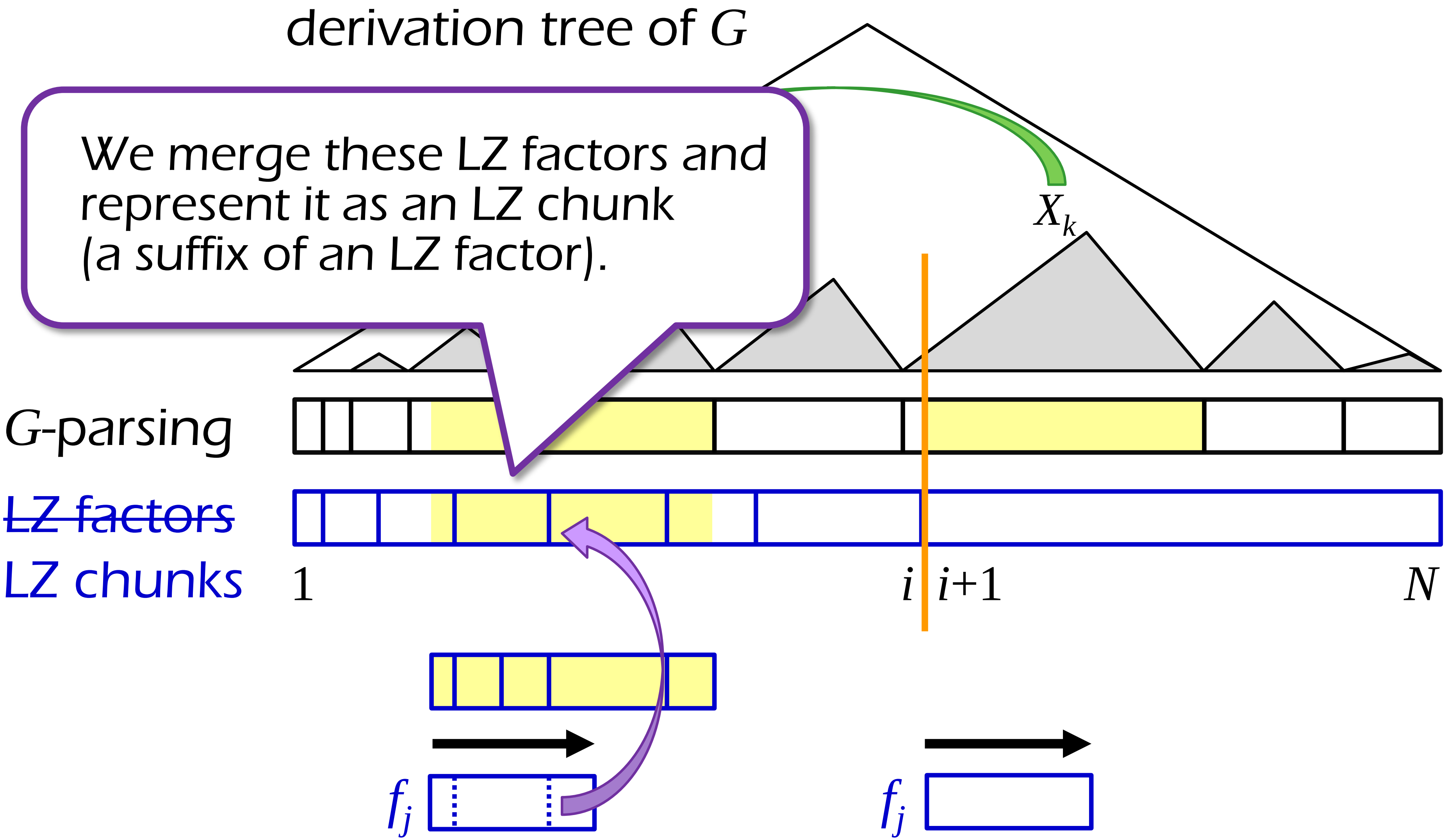
- ✓ Suppose we have computed LZ factors for $w[1..i]$.
- ✓ We compare $w[i+1..N]$ and previous LZ factors in BST.
- ✓ Each new LZ factor requires at most $\log m$ comparisons.

How do we compare previous LZ factor f_j and $w[i+1..N]$?

LZ78 Factorization on SLP [Cont.]



LZ78 Factorization on SLP [Cont.]



LZ78 Factorization on SLP [Cont.]

- ✓ Towards an $O(n + m \log m)$ bound:
  - ➢ Each LZ factor is computed by $O(\log m)$ comparisons to previous LZ factors.
  - ➢ The total number of LZ chunks that are involved in comparisons is $O(m \log m)$.
  - ➢ So, what remains is how to perform LCP query for two any LZ chunks in $\underline{O(1) \text{ time}}$!

LCP of Chunks

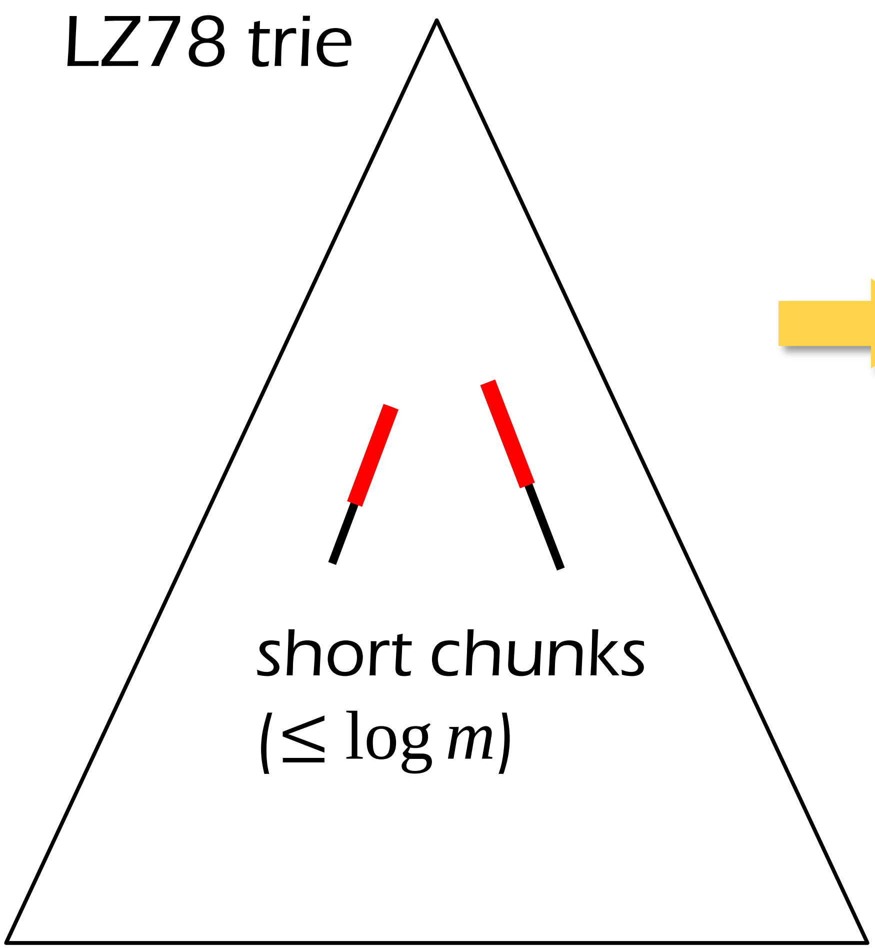
- ✓ Divide chunks into three types:
  1. Short chunks of length $\leq \log m$
  2. Medium chunks of length $\leq \log^2 m$
  3. Long chunks of length $> \log^2 m$
- ✓ Maintain a dynamic LCP data structure for each of them.

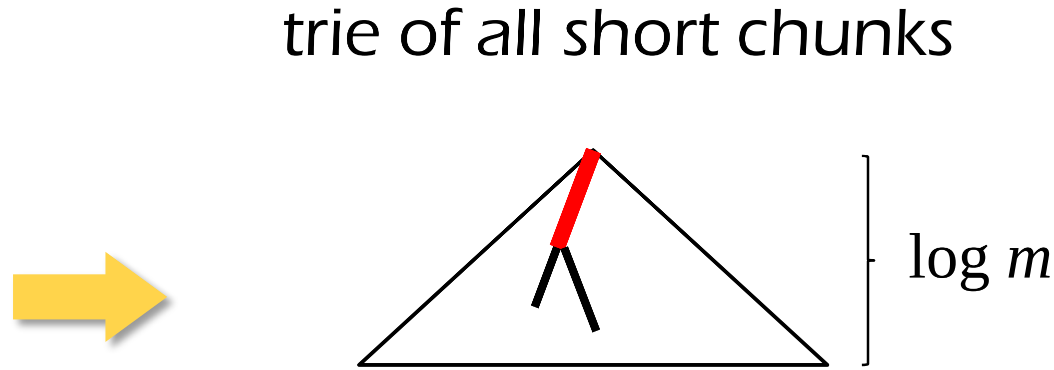
LCP of Short Chunks

Lemma 2

We can maintain a structure of size $O(m \log m)$ which supports updates in $O(\log m)$ time and LCP query of two short chunks ($\leq \log m$) in $O(1)$ time.

LCP of Short Chunks [Cont.]





$O(m \log m)$ nodes in this trie.

LCA gives LCP in $O(1)$ time.

A careful update procedure takes only $O(\log m)$ time, independently of alphabet.

LCP of Medium Chunks

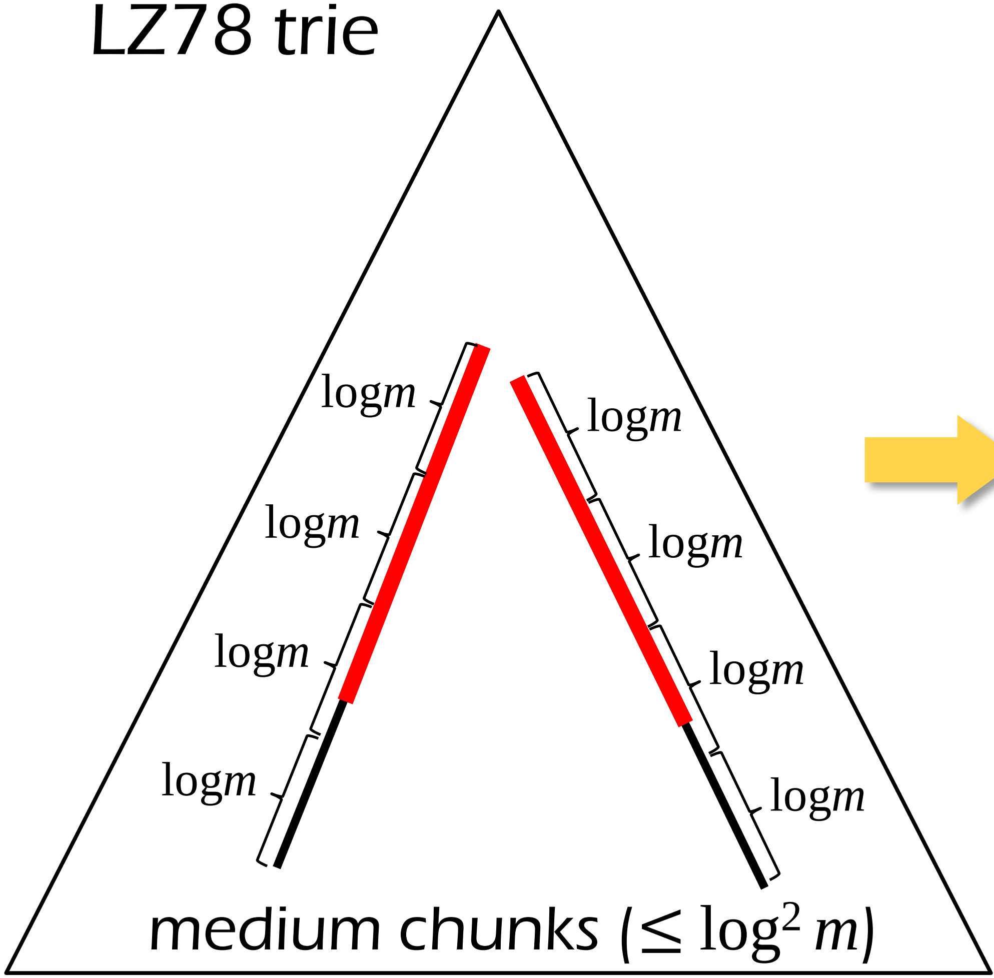
Lemma 3

We can maintain a structure of size $O(m \log m)$ which supports updates in $O(\log m)$ time and LCP query of two medium chunks ($\leq \log^2 m$) of length being multiple of $\log m$ in $O(1)$ time.

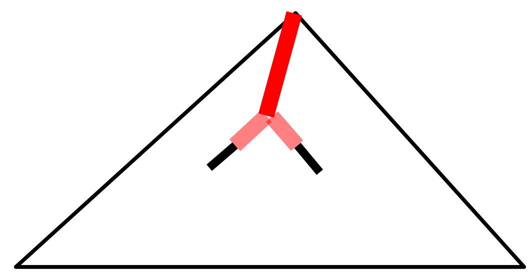
- ✓ The structure for short chunks works independently of the alphabet size.
- ✓ Regard a chunk of length $\log m$ as a meta-character, and use the structure for short chunks.

LCP of Medium Chunks [Cont.]

LZ78 trie



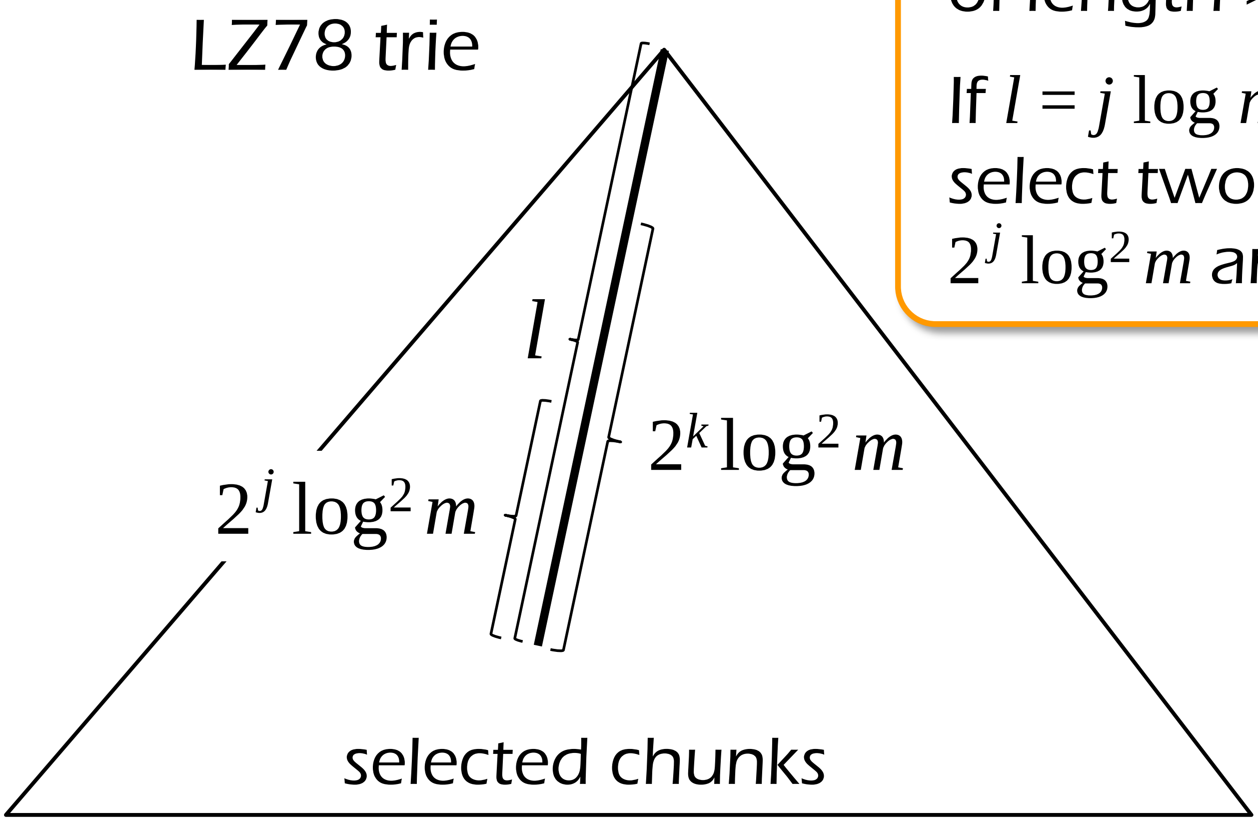
trie of short chunks
over meta-characters



LCA gives $\left\lfloor \frac{\mathrm{LCP}}{\log m} \right\rfloor$.

Last fragment can be computed by Lemma 2.

Towards LCP of Long Chunks



Let l be length of LZ factor of length $> \log^2 m$.

If $l = j \log m + k \pmod{\log^2 m}$, select two chunks of length $2^j \log^2 m$ and $2^k \log^2 m$.

Towards LCP of Long Chunks [Cont.]

Lemma 4

We can maintain a structure of size $O(m \log m)$ which supports updates in $O(\log m)$ time and LCP query of two selected chunks $(2^k \log^2 m)$ of the same length in $O(1)$ time.

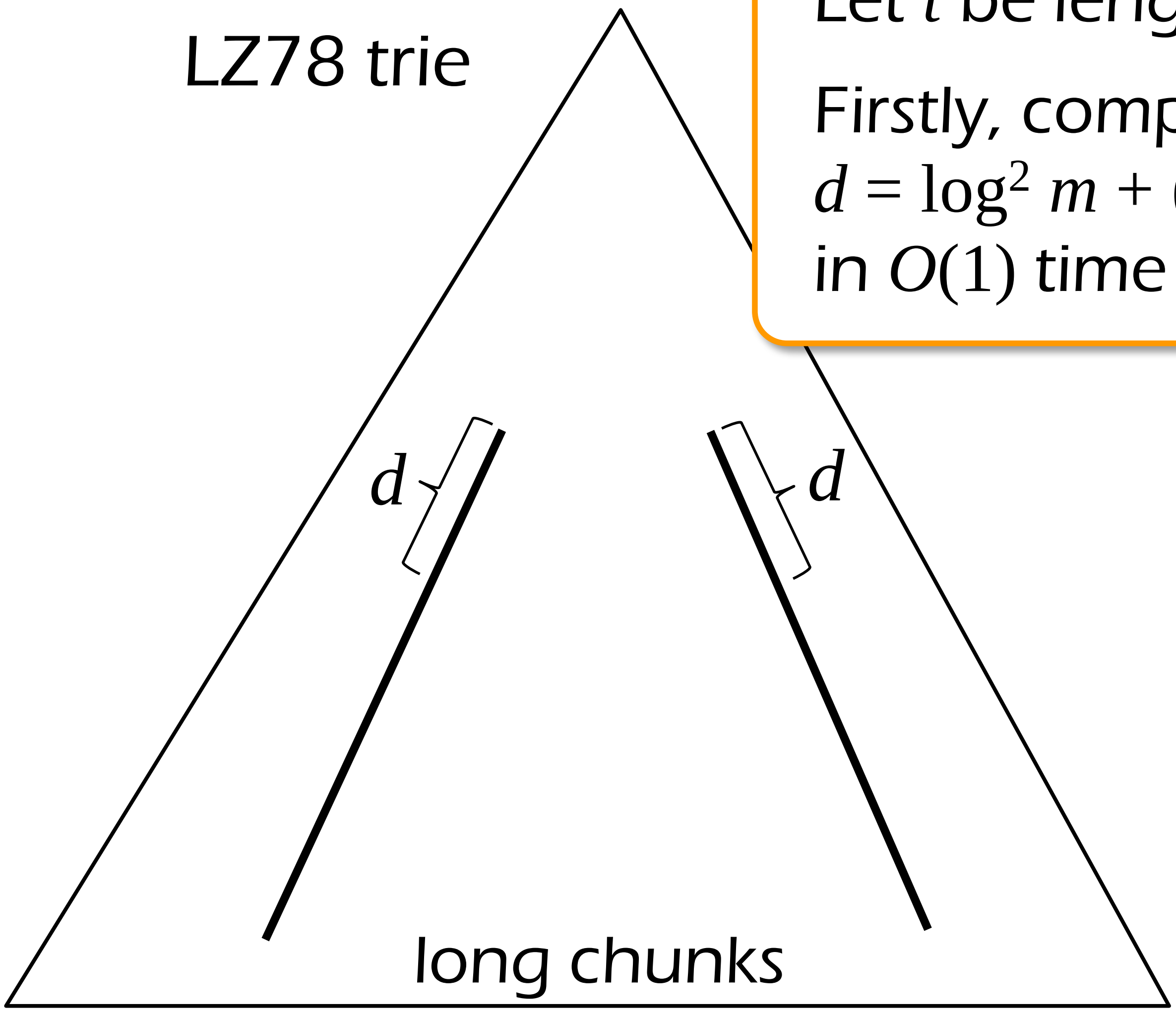
- ✓ Details are omitted here.
- ✓ Please see the paper for details.

LCP of Long Chunks

Lemma 5

We can maintain a structure of size $O(m \log m)$ which supports updates in $O(\log m)$ time and LCP query of two long chunks ($> \log^2 m$) in $O(1)$ time.

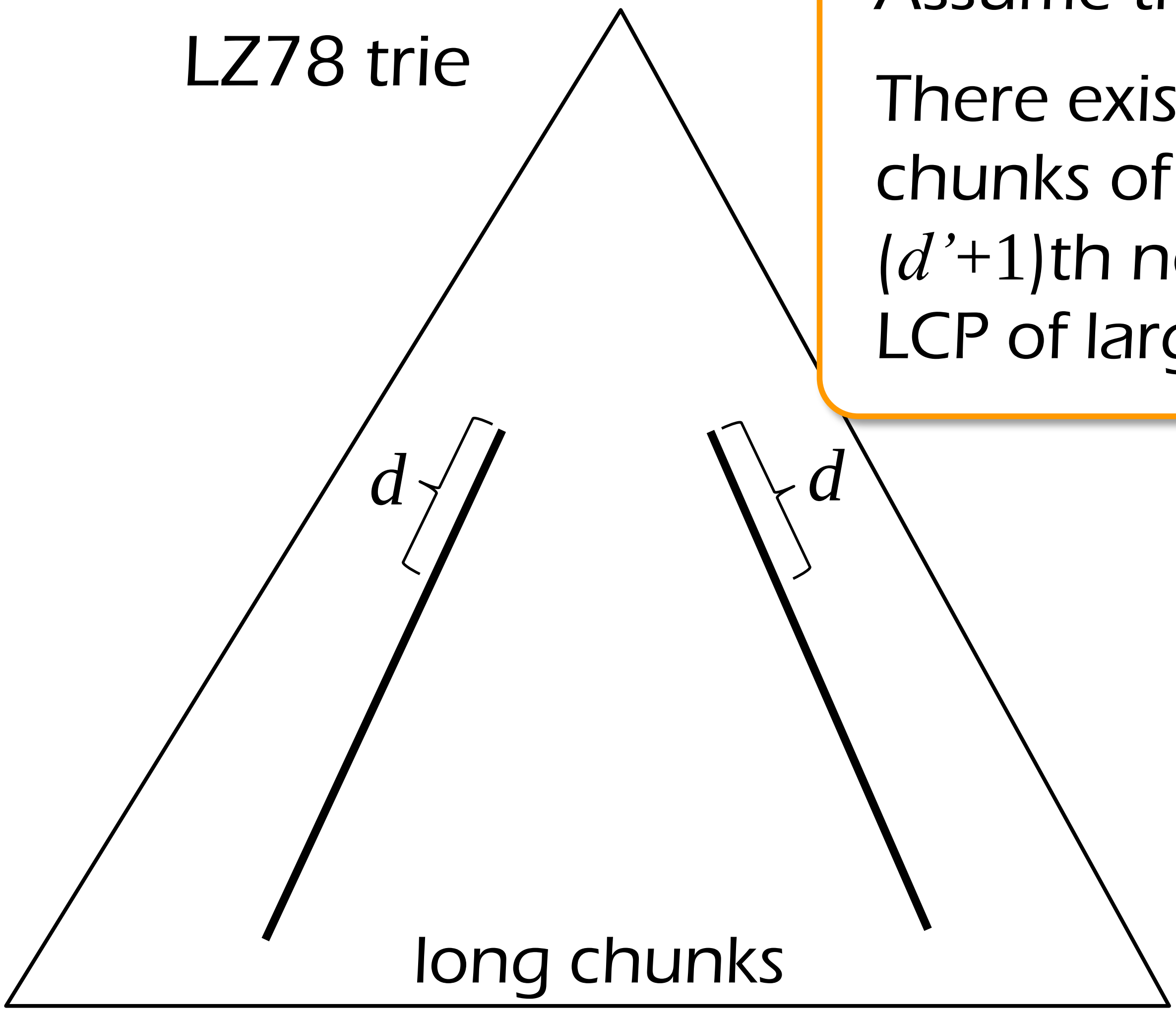
LCP of Long Chunks [Cont.]



Let t be length of chunks.

Firstly, compare prefixes of length
$d = \log^2 m + (t \mod \log^2 m)$
in $O(1)$ time by Lemmas 2 & 3.

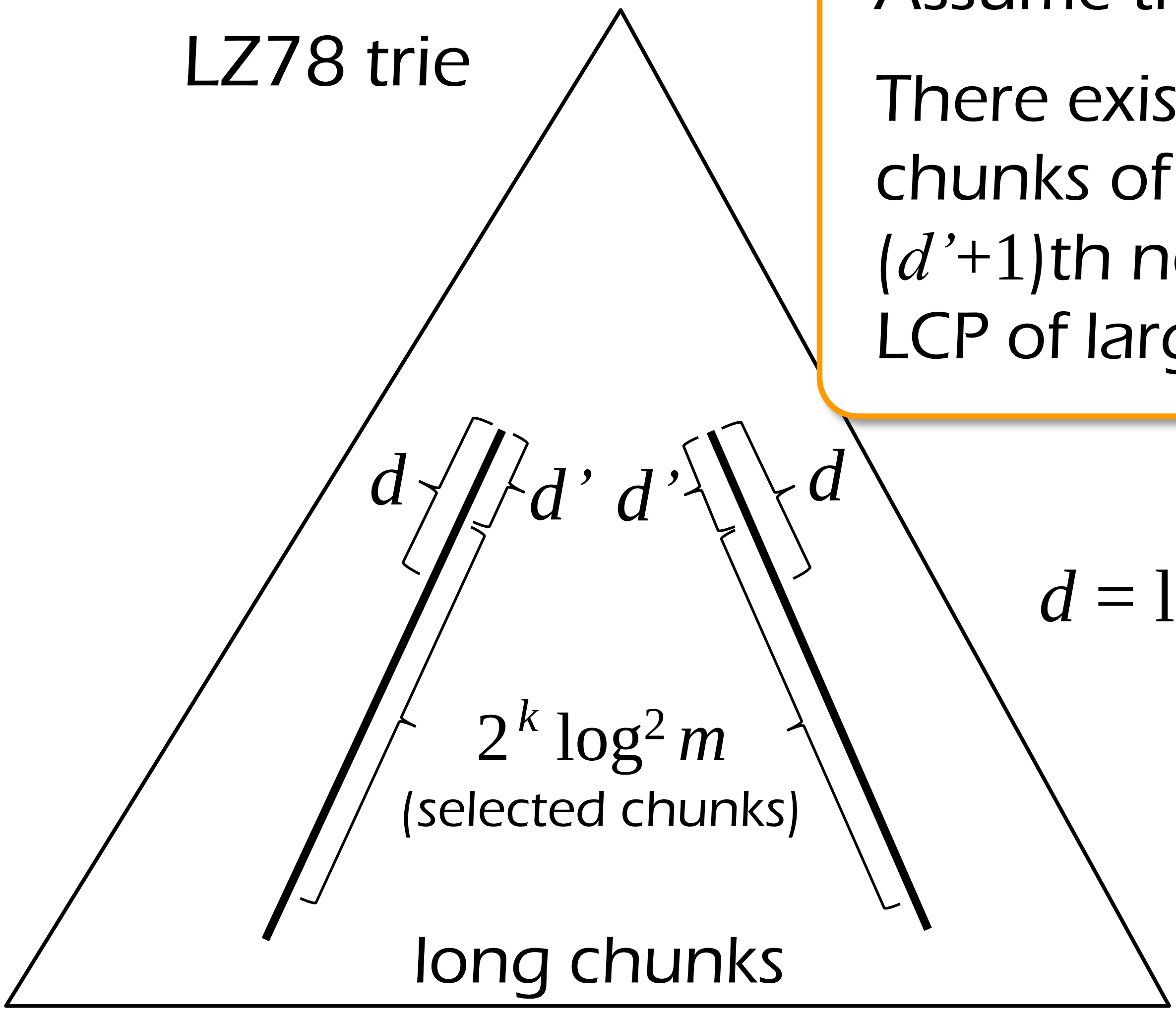
LCP of Long Chunks [Cont.]



Assume the prefixes are equal.

There exists $d' \leq d$ s.t. selected chunks of same length starts at $(d'+1)$th node, so we compute LCP of largest selected chunks.

LCP of Long Chunks [Cont.]



Assume the prefixes are equal.

There exists $d' \leq d$ s.t. selected chunks of same length starts at $(d'+1)$th node, so we compute LCP of largest selected chunks.

$$d = \log^2 m + (t \bmod \log^2 m)$$

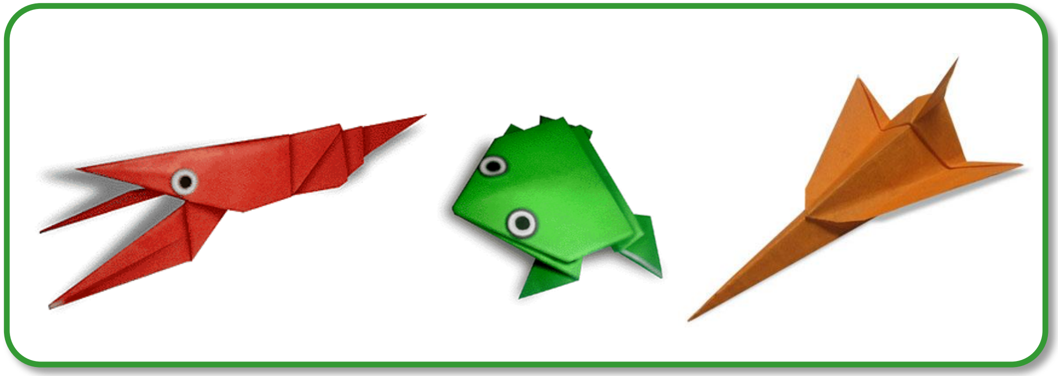
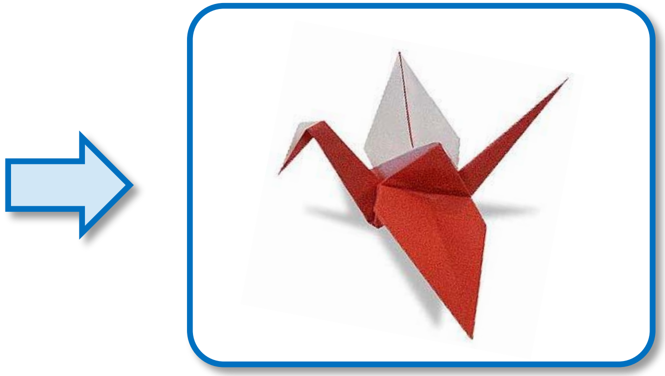
Dynamic $O(1)$-time Chunk LCP

Theorem 3

For a growing trie with at most m nodes, we can maintain a structure of size $O(m \log m)$ which supports updates in $O(\log m)$ time and LCP query for two any chunks (i.e., any paths) in $O(1)$ time.

Concluding Remarks

- ✓ We proposed an $O(n + m \log m)$ time & space algorithm which converts any SLP to LZ78.
- ✓ An $O(n + m)$ space & $O((n + m) \log m)$ time variant exists.
- ✓ The dynamic LCP data structure can be used for <u>any growing trie</u>.

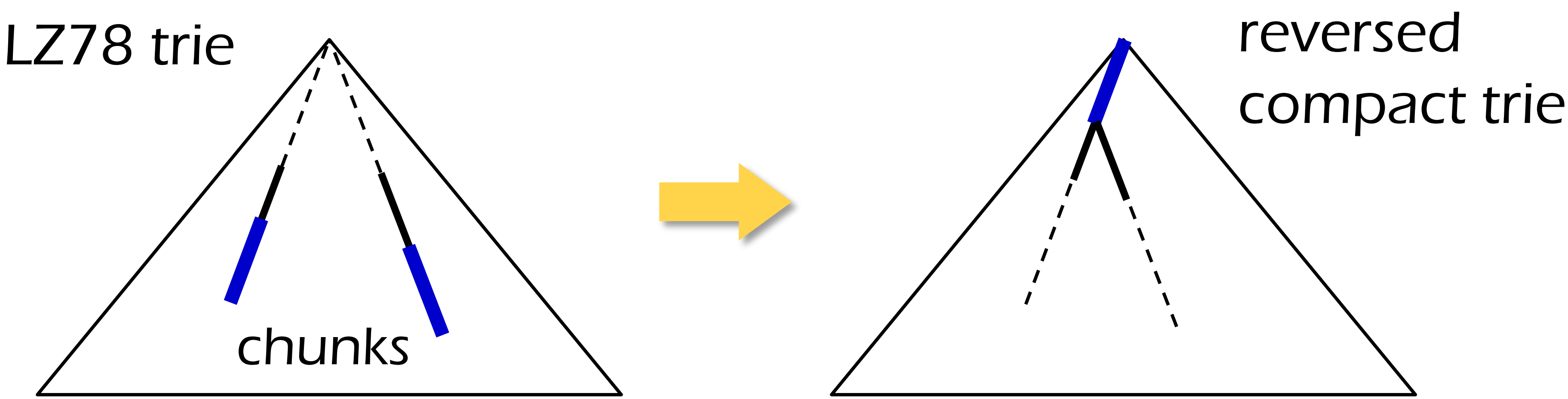



Appendices

Longest Common Suffix of Chunks

Lemma 2

We can maintain a structure of size $O(m)$ which supports updates in $O(\log m)$ time and LCS query of any two chunks in $O(1)$ time.



Dynamic LCA
[Cole & Hariharan, 1999]

Longest Common Suffix of Chunks [Cont.]

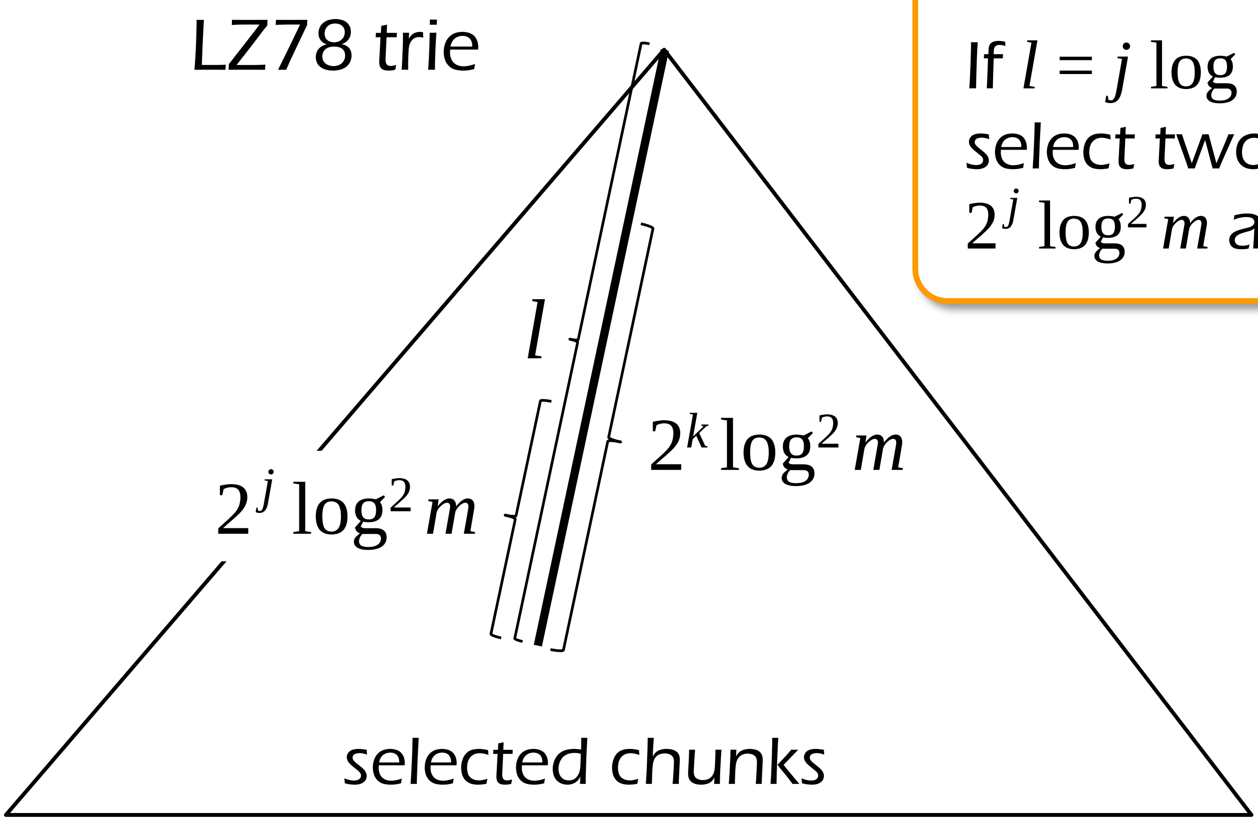
Lemma 2

We can maintain a structure of size $O(m)$ which supports updates in $O(\log m)$ time and LCS query of any two chunks in $O(1)$ time.

✓ A new reversed LZ factor can be inserted in $O(\log m)$ time (independently of the alphabet size).

- BST for the reversed LZ factors;
- Dynamic LCA [Cole & Hariharan, 1999]
- Dynamic level ancestor [Alstrup & Holm, 2000];

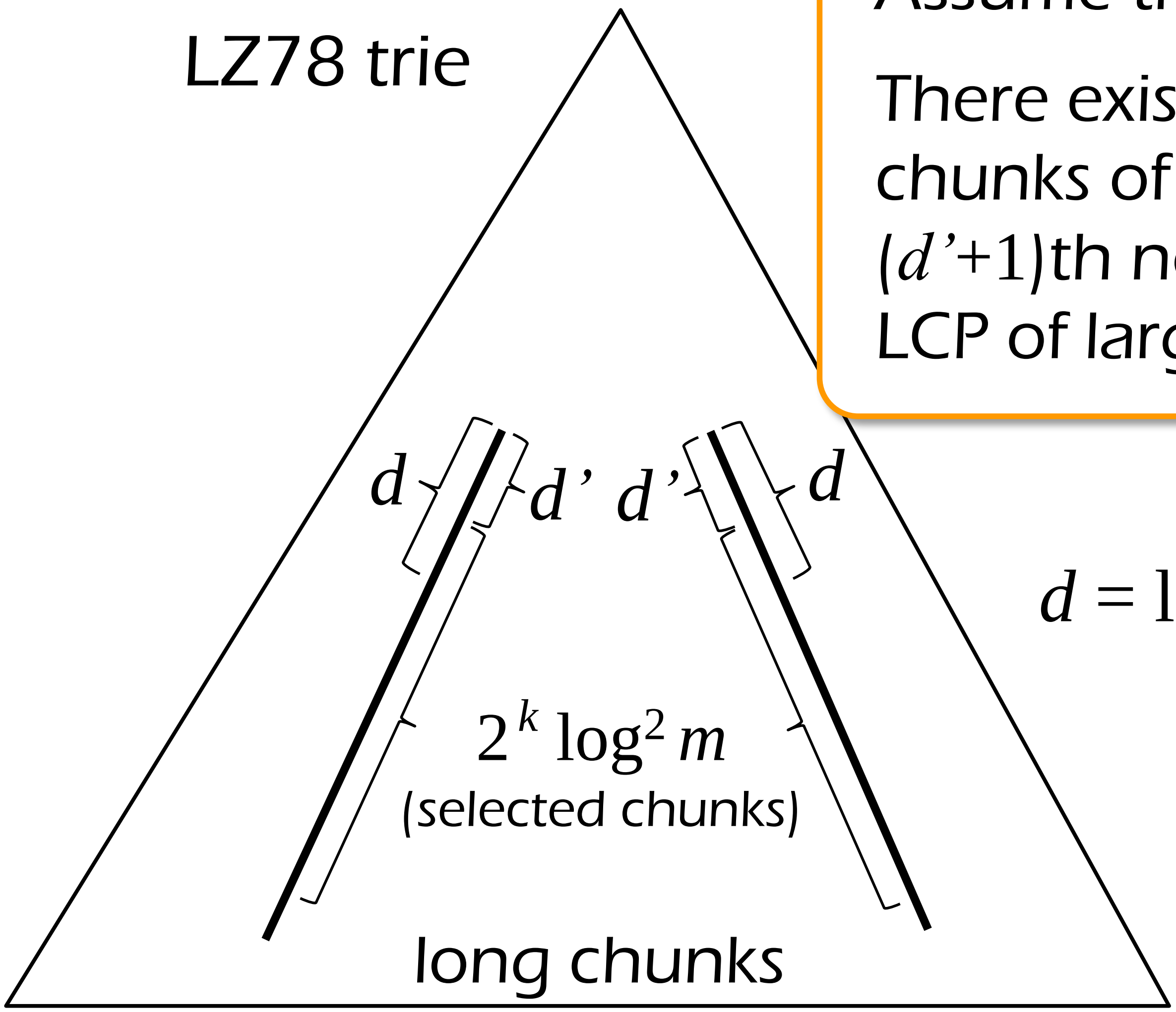
Selected Chunks



Let l be length of LZ factor.

If $l = j \log m + k \pmod{\log^2 m}$, select two chunks of length $2^j \log^2 m$ and $2^k \log^2 m$.

LCP of Long Chunks [Cont.]



Assume the prefixes are equal.

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$$d = \log^2 m + (t \bmod \log^2 m)$$