CPM 2016

# Factorizing a string into squares in linear time

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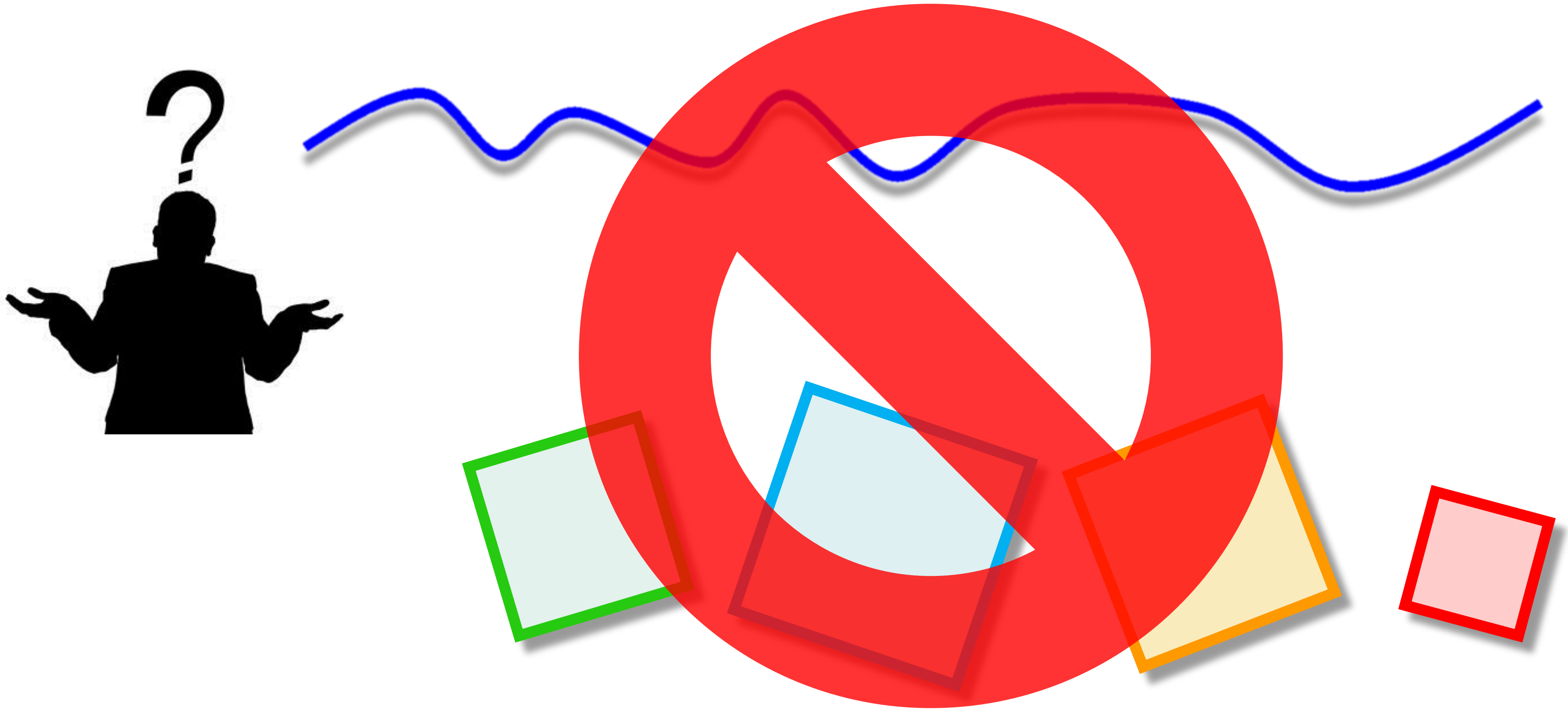
# From string to squares?

- In this presentation, I talk about decomposition of a *string* into *squares*.



# Squares (as strings!)

- "Our square" is a string of form $xx$.
  - aabaab
  - abababab
  - ababaababa

# Primitively rooted squares

- A square $xx$ is called a *primitively rooted square* if its root $x$ is primitive (i.e., $x \neq y^k$ for any string $y$ and integer $k$).
  - **aabaab** : primitively rooted square
  - **abababab** : not primitively rooted square
  - **ababaababa** : primitively rooted square

# Our problem

- Determine whether a given string can be factorized into a sequence of squares.
  If the answer is yes, then compute one of such factorizations.

E.g.)

- aabaabaaaaaa → Yes
  - (aabaab, aaaaaa),
  - (aabaab, aaaa, aa),
  - (aa, baabaa, aa, aa), and so on.
- aabaabbbab → No

# Previous work

Times for computing square factorization

| | [Dumitran et al., 2015] |
|---|---|
| A sq. factor. | $O(n \log n)$ |

- $n$ is the length of the input string.

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| | [Dumitran et al., 2015] | Our solutions |
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| A sq. factor. | $O(n \log n)$ | $O(n)$ |
| Largest sq. factor. | $O(n \log n)$ | $O(n + (n \log^2 n) / \omega)$ |
| Smallest sq. factor. | – | $O(n \log n)$ |

- $n$ is the length of the input string.
- Our results for arbitrary/largest square factorizations are valid on word RAM with word size $\omega = \Omega(\log n)$.

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# Simple observation

- Every square is of even length.
- Thus, if string $w$ has a square factorization, then $w$ also has a square factorization which consists *only of primitively rooted squares*.

E.g.)

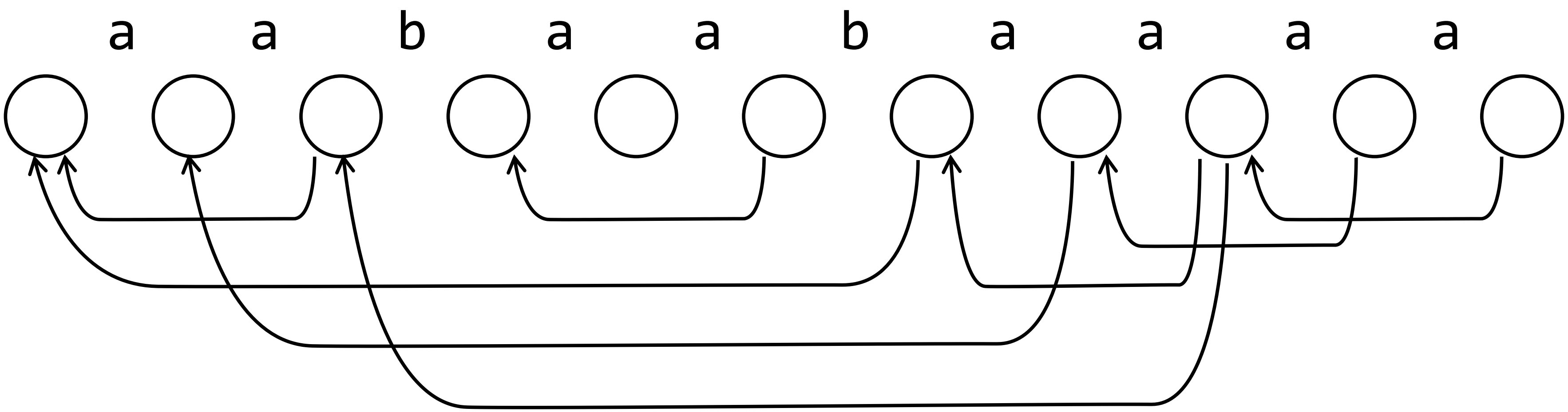
- aaaaaa|abababab
- aa|aa|aa|abab|abab

# # of primitively rooted squares

- Any string of length $n$ contains $O(n \log n)$ primitively rooted squares [Crochemore & Rytter, 1995].
- The simple observation + the above lemma lead to a natural DP approach which computes a square factorization in $O(n \log n)$ time.

# Dumitran et al.'s algorithm

- Consider the following DAG $G$ for string $w$:
  - There are $n$+1 nodes.
  - There is a directed edge $(e+1, b)$ in $G$. $\Leftrightarrow$ Substring $w[b..e]$ is a primitively rooted square.



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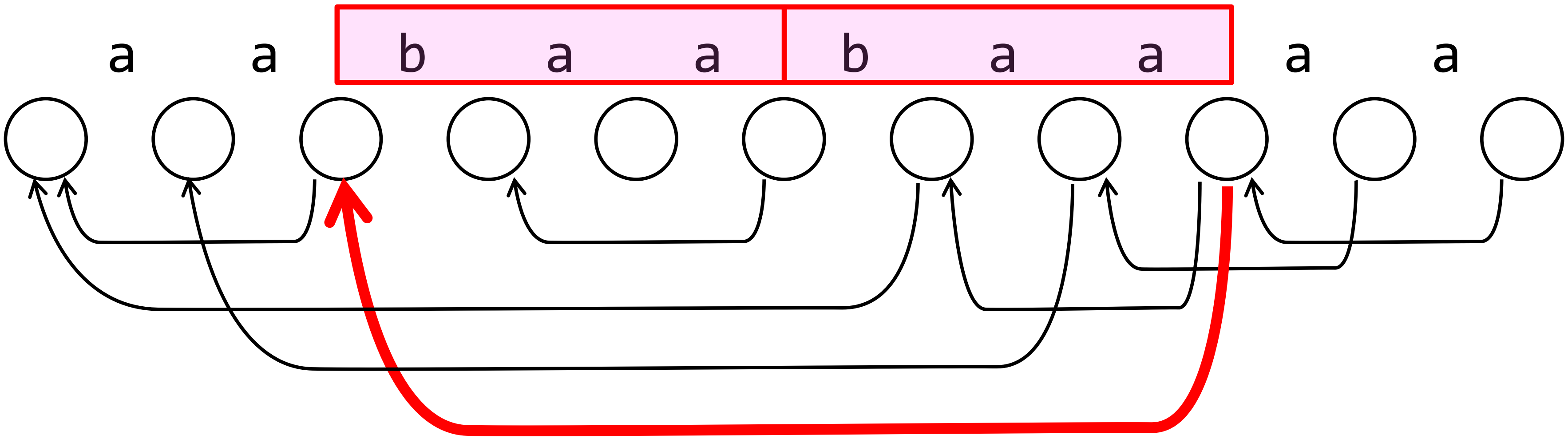
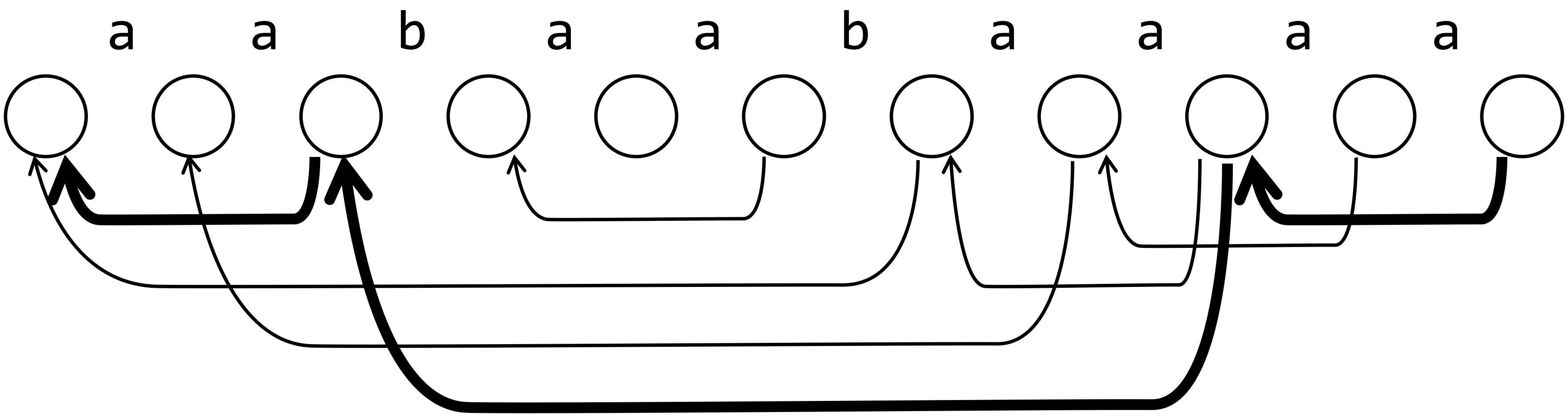


# Dumitran et al.'s algorithm

- DAG $G$ has a path from the rightmost node to the leftmost node.
  $\Leftrightarrow$ There is a square factorization of $w$.



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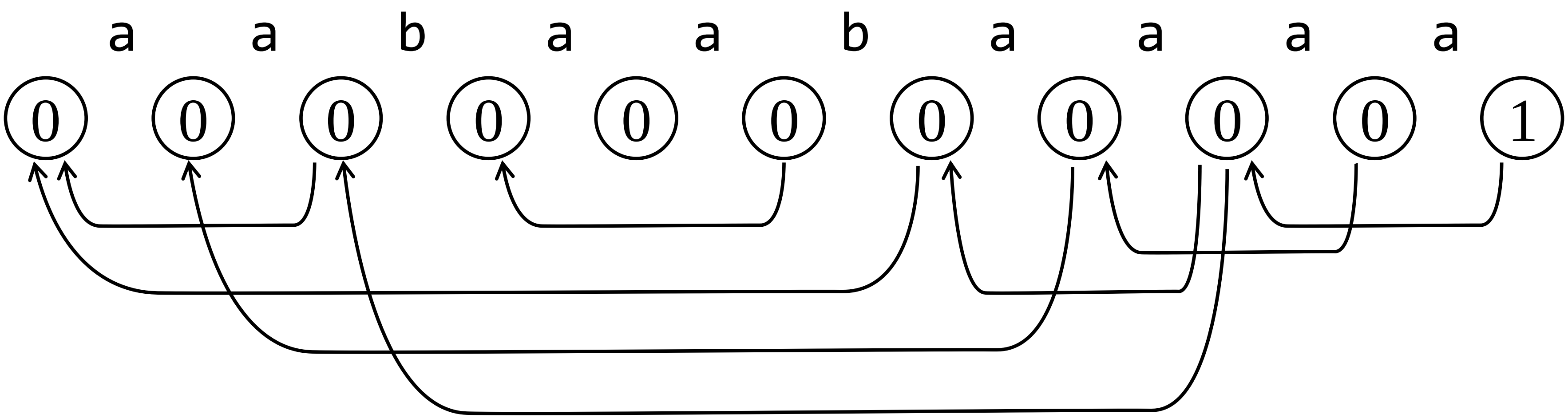
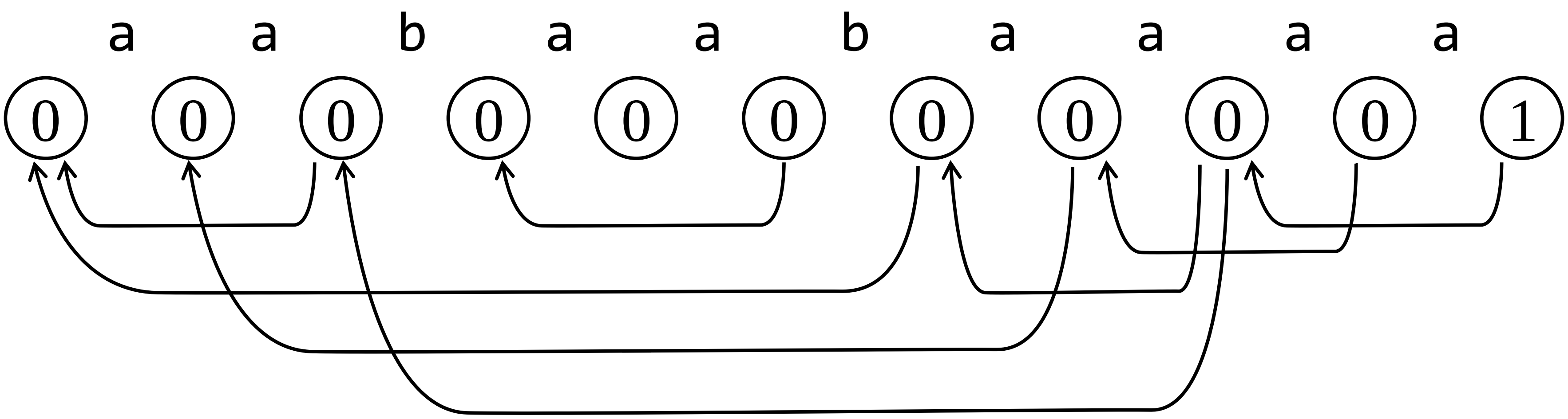



- The rightmost node is associated with a $1$.
- Initially, all the other nodes are associated with $0$'s.

# Dumitran et al.'s algorithm




- We process each node from right to left.
- Each node $v$ gets a $1$ iff there is an in-coming edge to $v$ from a node that is associated with a $1$.

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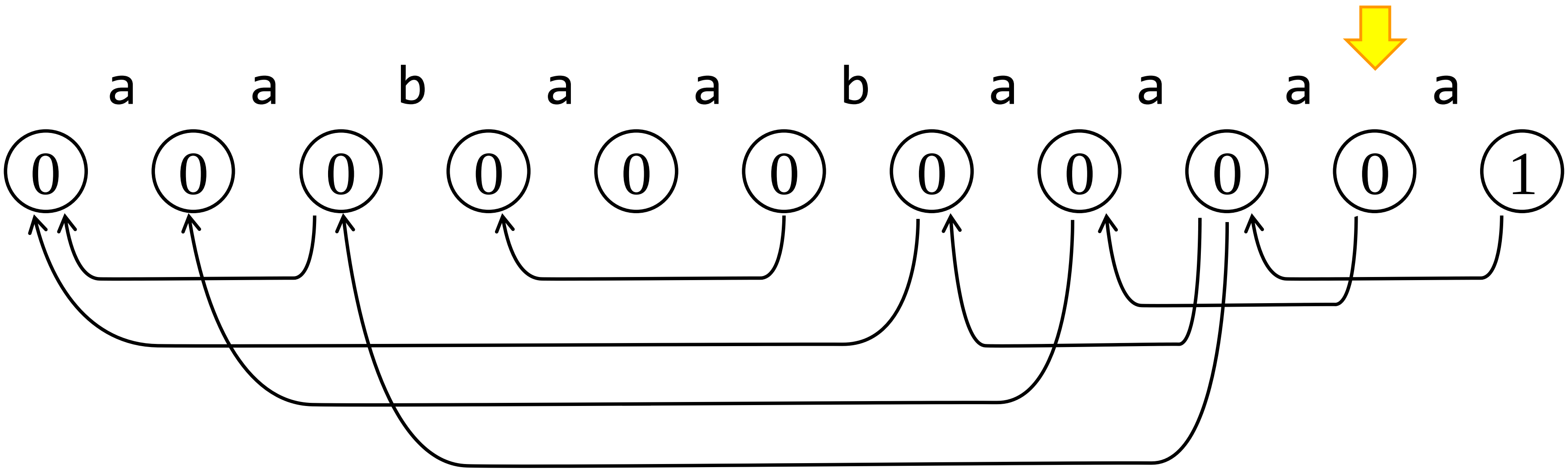


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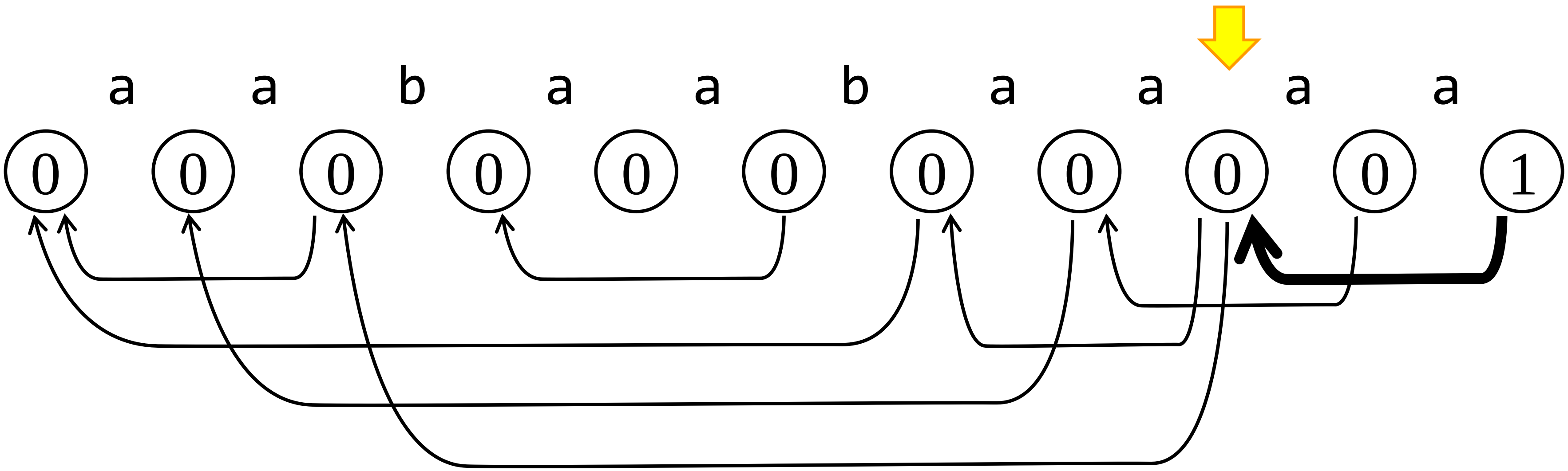


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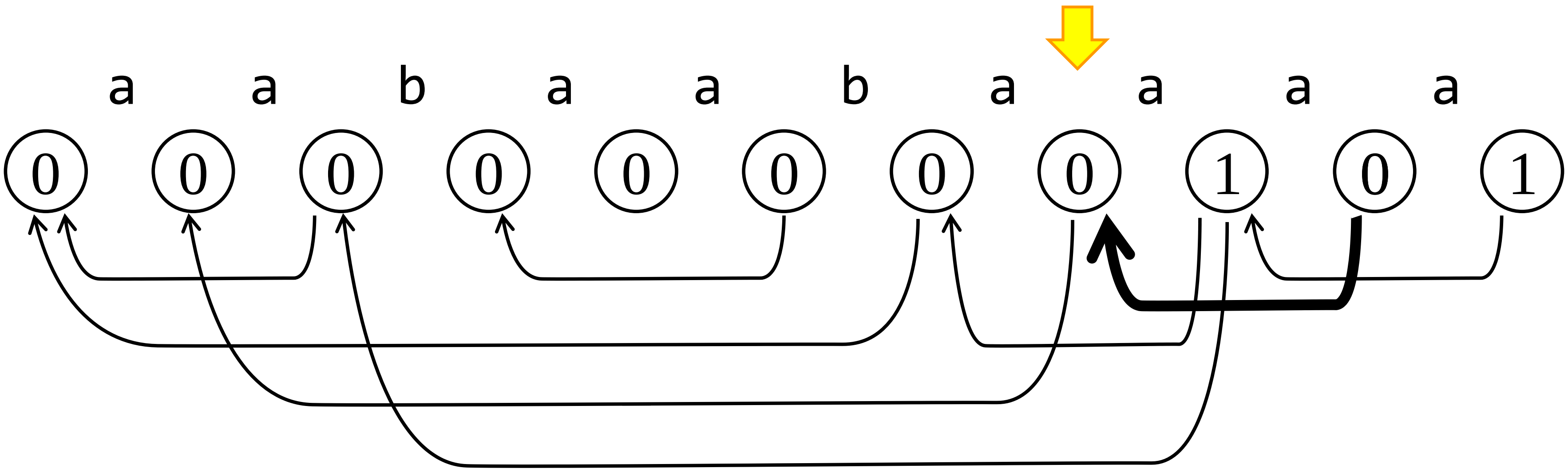


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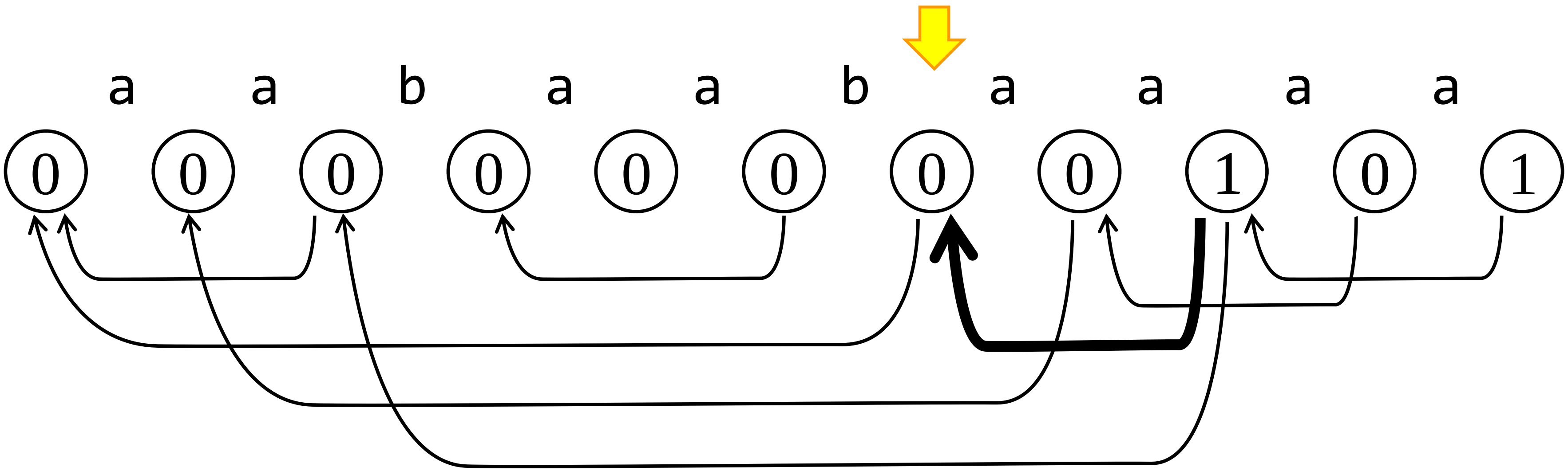


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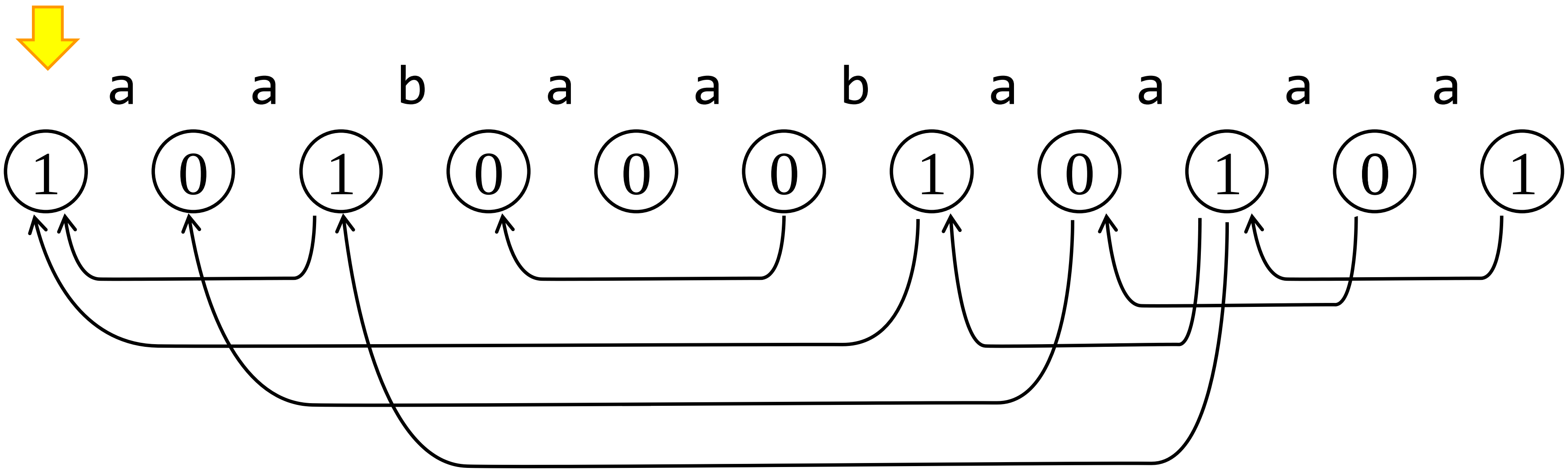
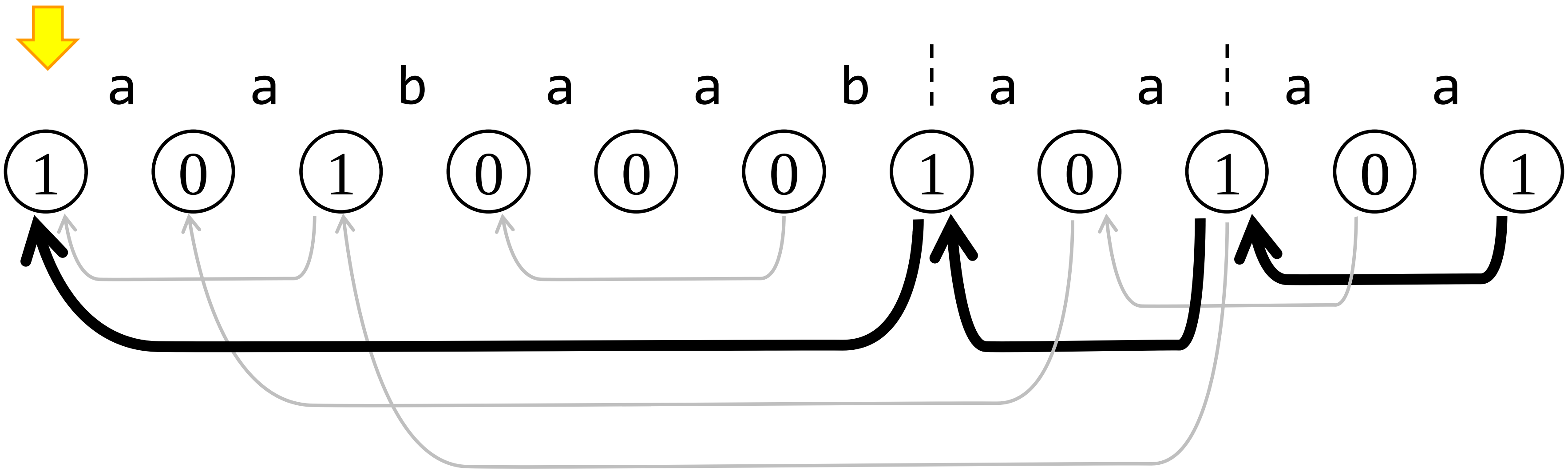


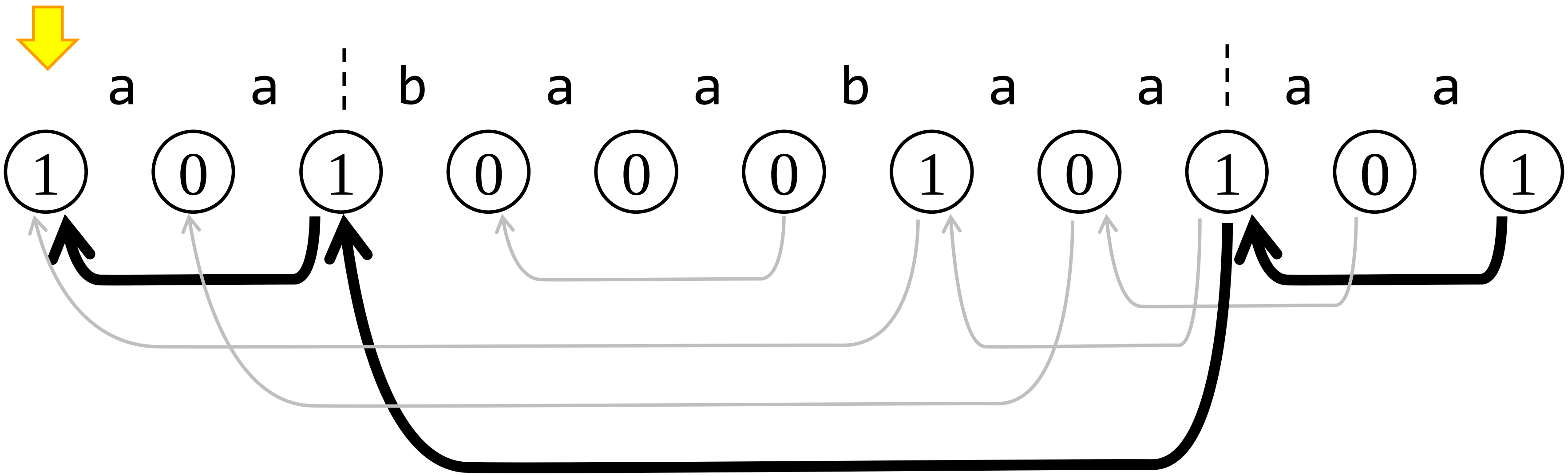

- Finally, there is a square factorization of the string iff the leftmost node is associated with a 1.

# Dumitran et al.'s algorithm



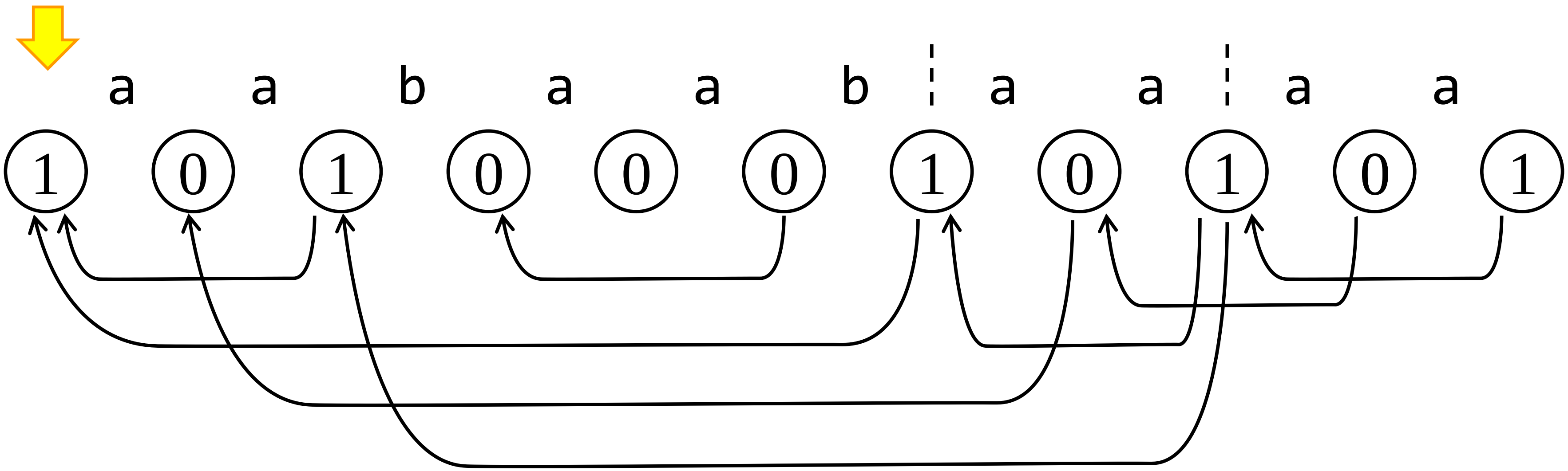

- A path from the rightmost node to the leftmost node corresponds to a square factorization.

# Dumitran et al.'s algorithm




- Another path from the rightmost node to the leftmost node corresponds to another square factorization.

# Dumitran et al.'s algorithm



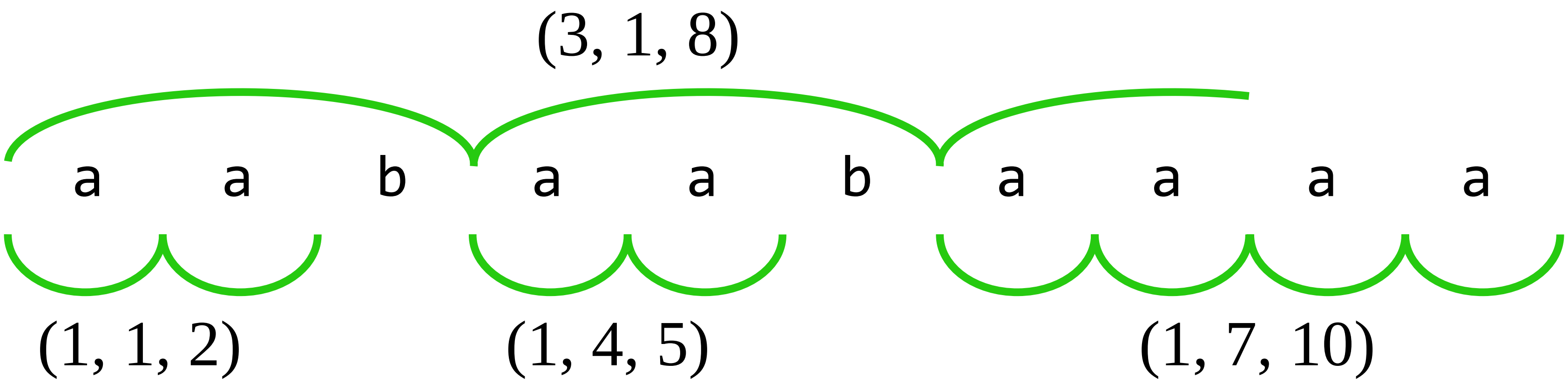

- Clearly, the number of edges in this DAG is equal to the number of primitively rooted squares in the string, which is $O(n \log n)$ .
- Hence, their algorithm takes $O(n \log n)$ time.

# Ideas of our $O(n)$-time algorithm

- We accelerate Dumitran et al.'s algorithm by a mixed use of
  - ***runs*** (maximal repetitions in the string);
  - ***bit parallelism*** (performing some DP computation in a batch).

# Runs

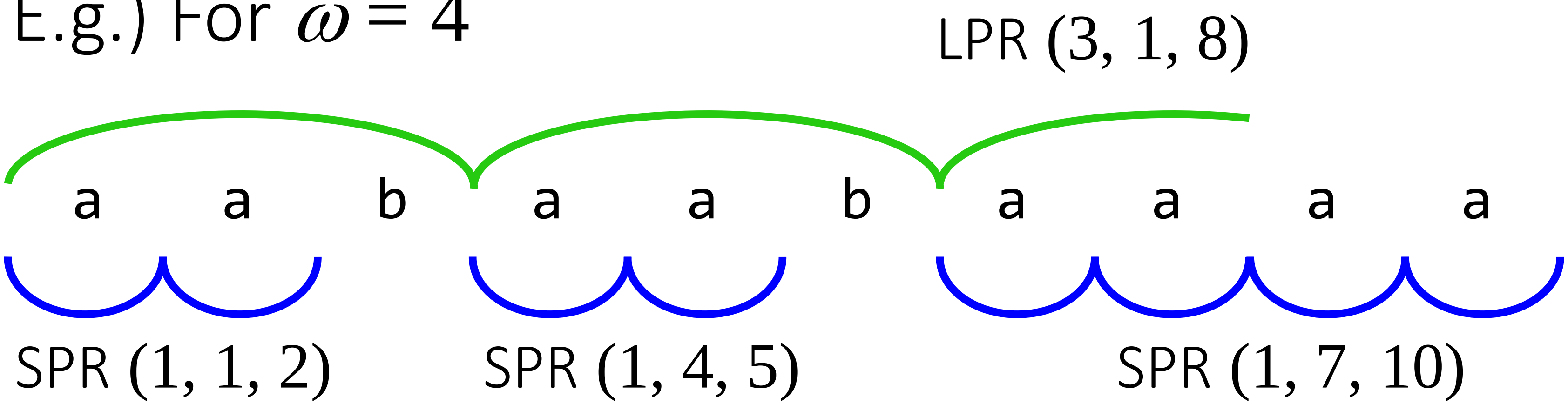
- A triple $(p, b, e)$ of integers is said to be a *run* of a string $w$ if
  - The substring $w[b..e]$ is a repetition with the smallest period $p$ (i.e., $2p \leq e-s+1$), and
  - The repetition is non-extensible to left nor right with the same period $p$.



# Long and short period runs

- Let $\omega$ be the machine word size.
- A run $(p, b, e)$ in a string is called
  - a *long period run* (*LPR*) if $2p \geq \omega$ ;
  - a *short period run* (*SPR*) if $2p < \omega$ .

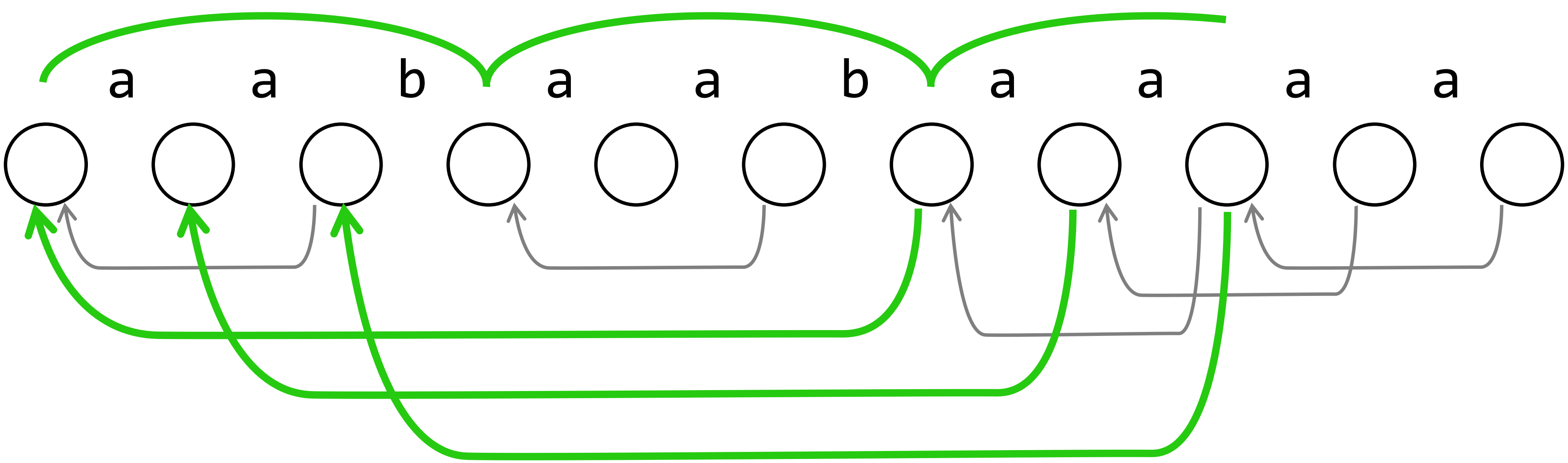
E.g.) For $\omega = 4$



# Long edges

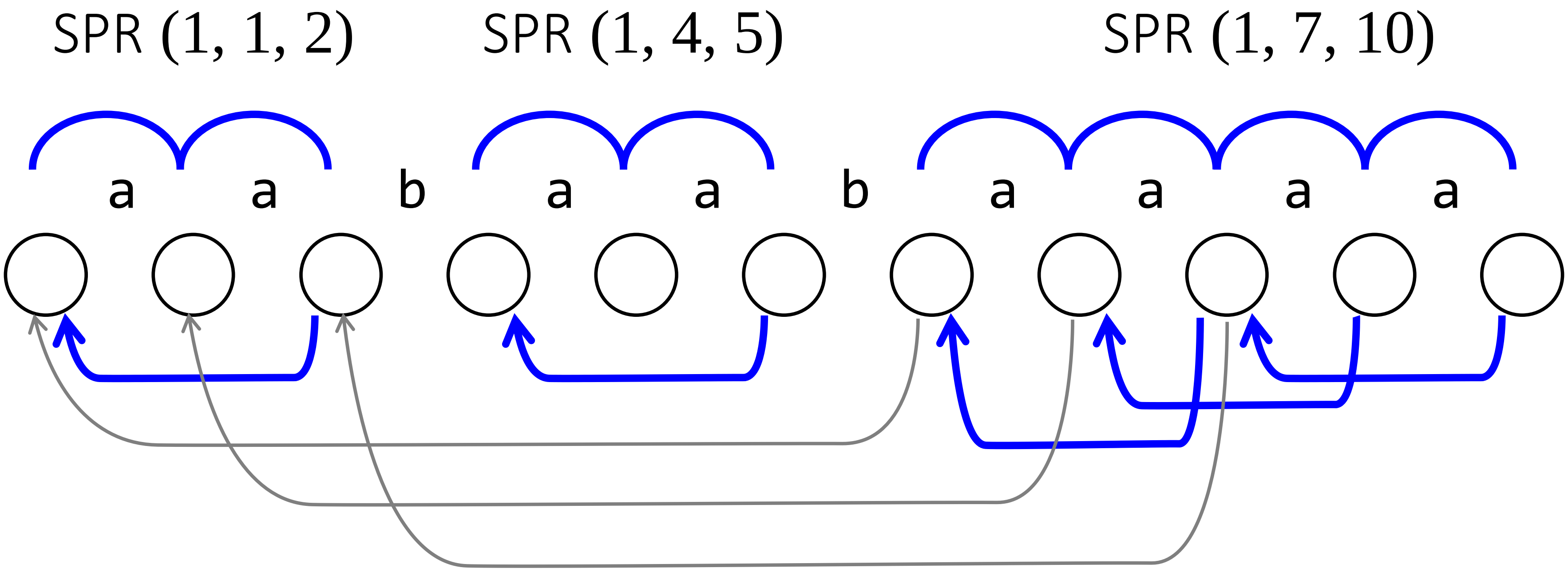
- Edges that correspond to long period runs are called *long edges.*

LPR $(3, 1, 8)$



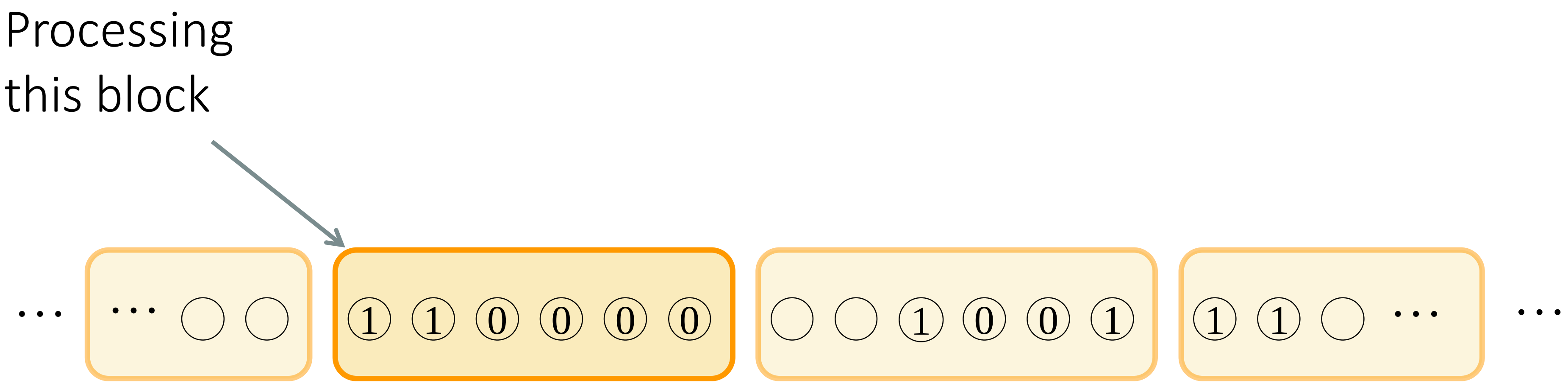
# Short edges

- Edges that correspond to short period runs are called *short edges.*



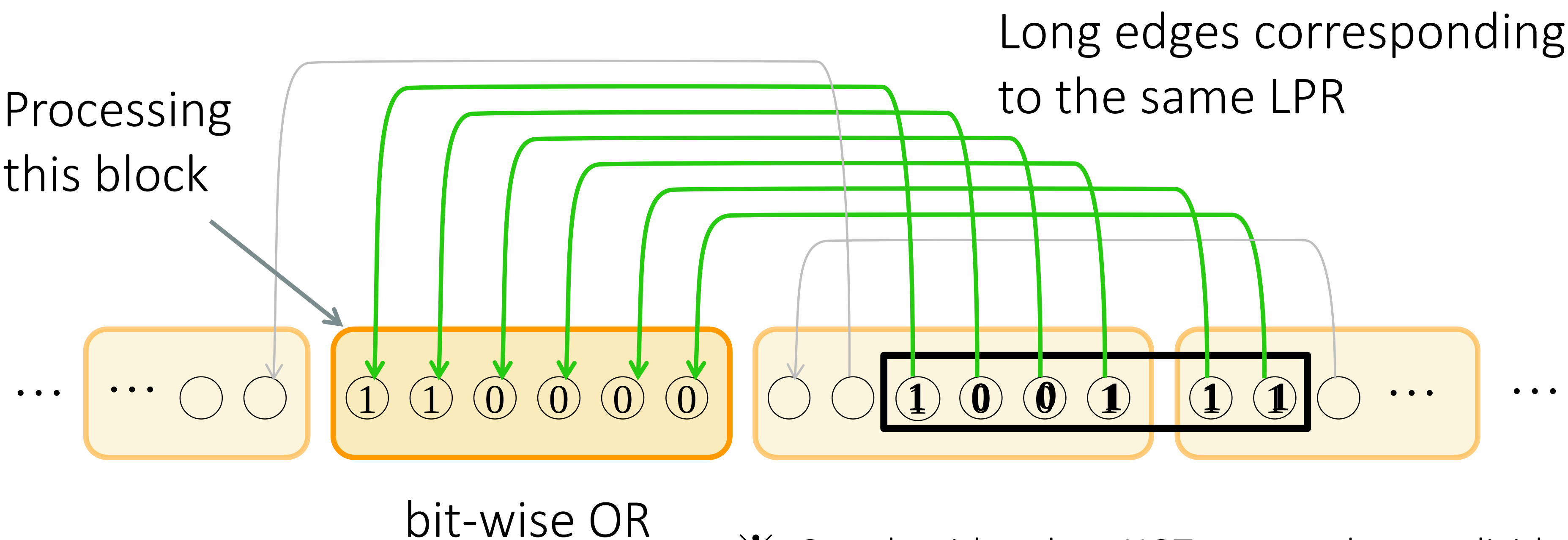
# How to process long edges

- We partition the nodes into blocks of length $\omega$ each.



# How to process long edges

- Since the long edges that correspond to the same LPR have the same length and are consecutive, we can process $\omega$ of them in a batch, by performing a bit-wise OR.

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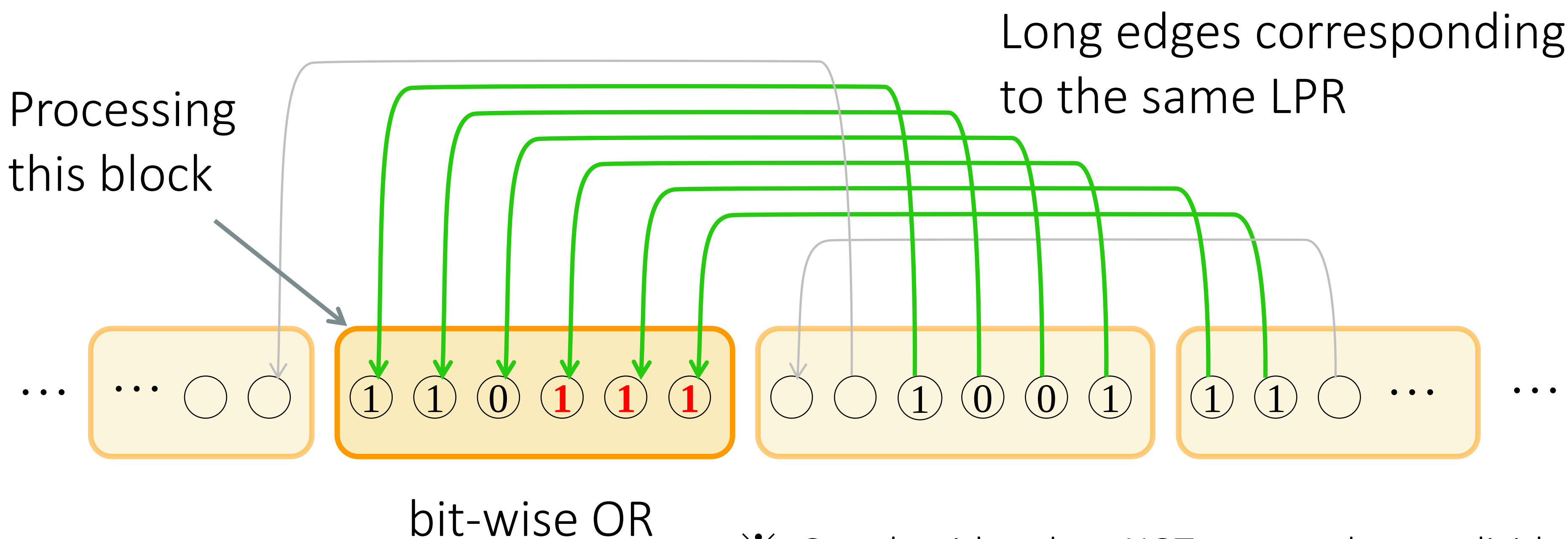


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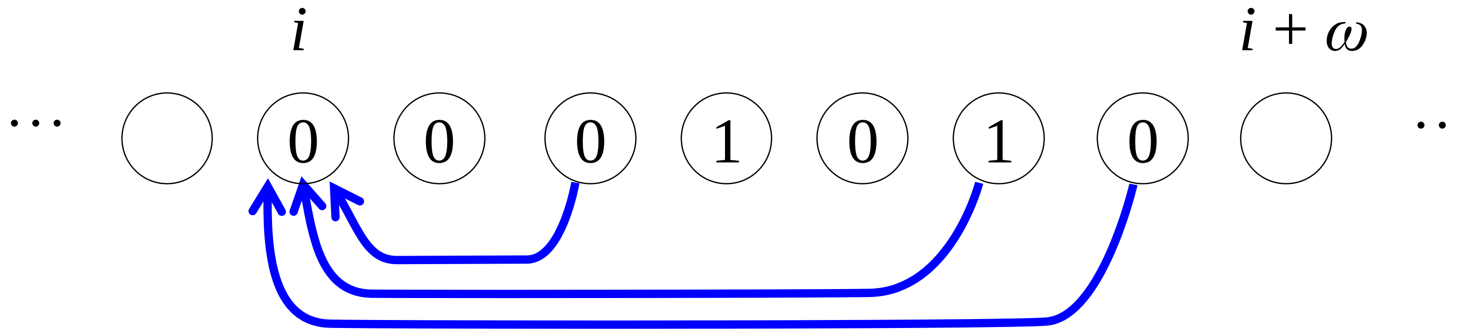
# Time cost for long edges

- We can process at most $\omega$ long edges in a batch in $O(1)$ time, hence we can process all long edges in $O((n \log n)/\omega)$ time.
- An $O(n + \#\text{LPR})$-time preprocessing allows us to perform the these operations without constructing long edges explicitly.
- Thus we need $O(n + \#\text{LPR} + (n \log n)/\omega)$ total time for long edges.

# How to process short edges

- Every short edge is shorter than $\omega$ .
- Hence, for each node $i$, it is enough to consider at most $\omega$ in-coming short edges.

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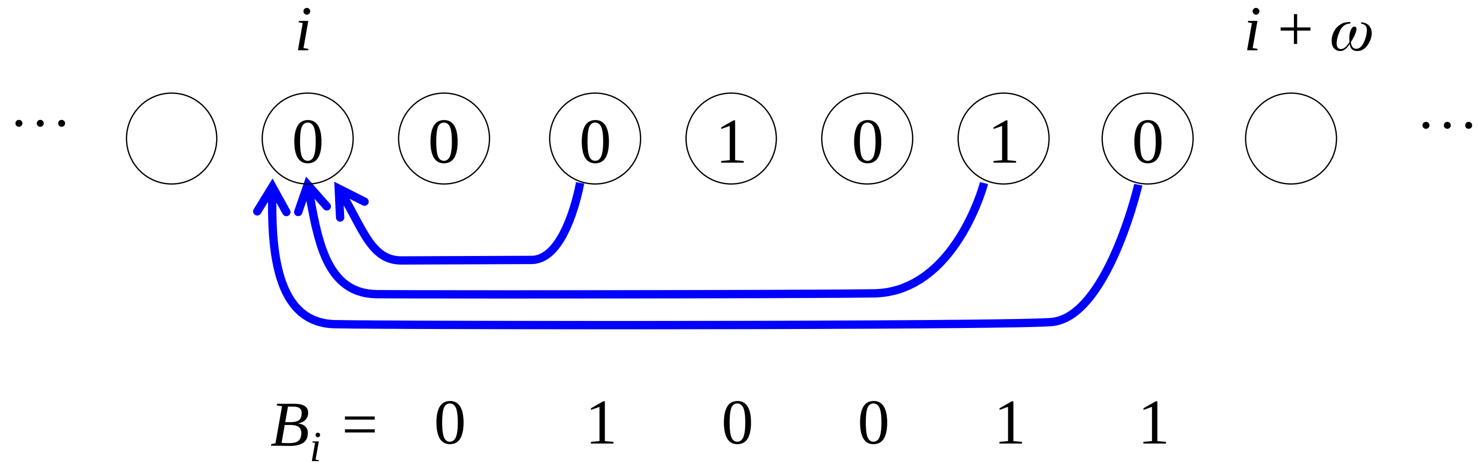
# How to process short edges

- To process these short edges in a batch, we use a bit mask $B_i$ indicating if each node has a short edge to node $i$.

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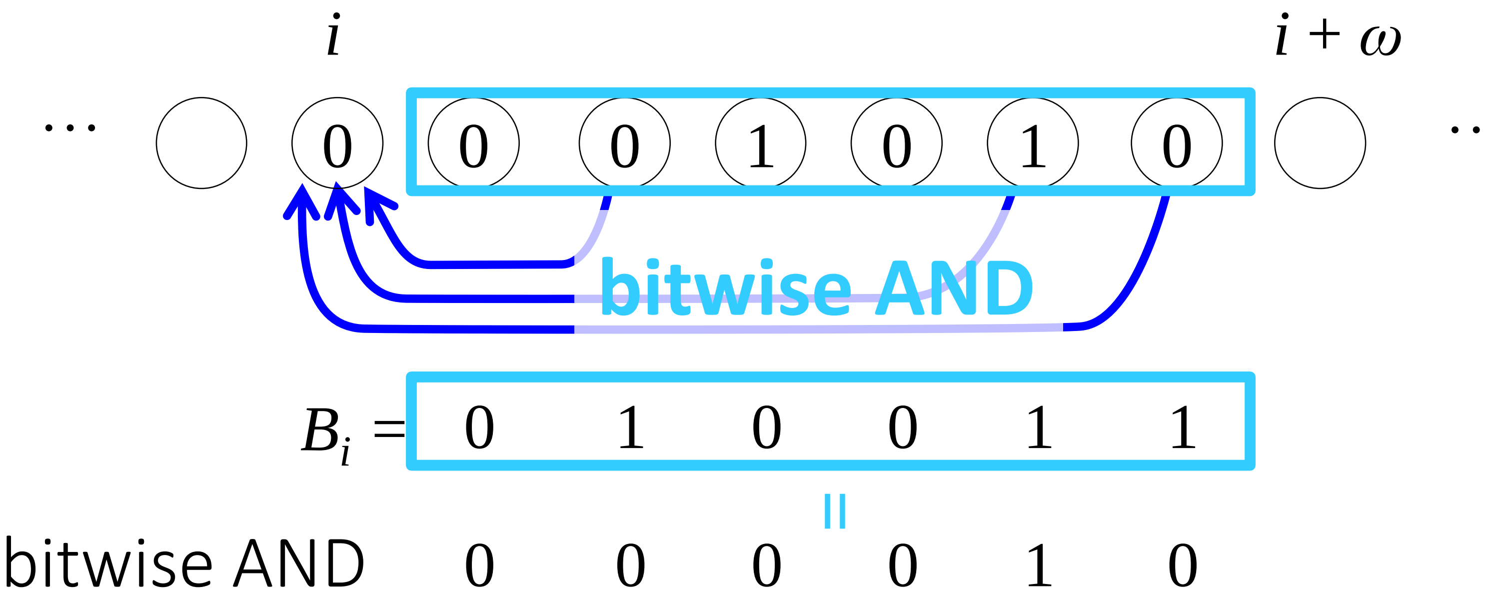


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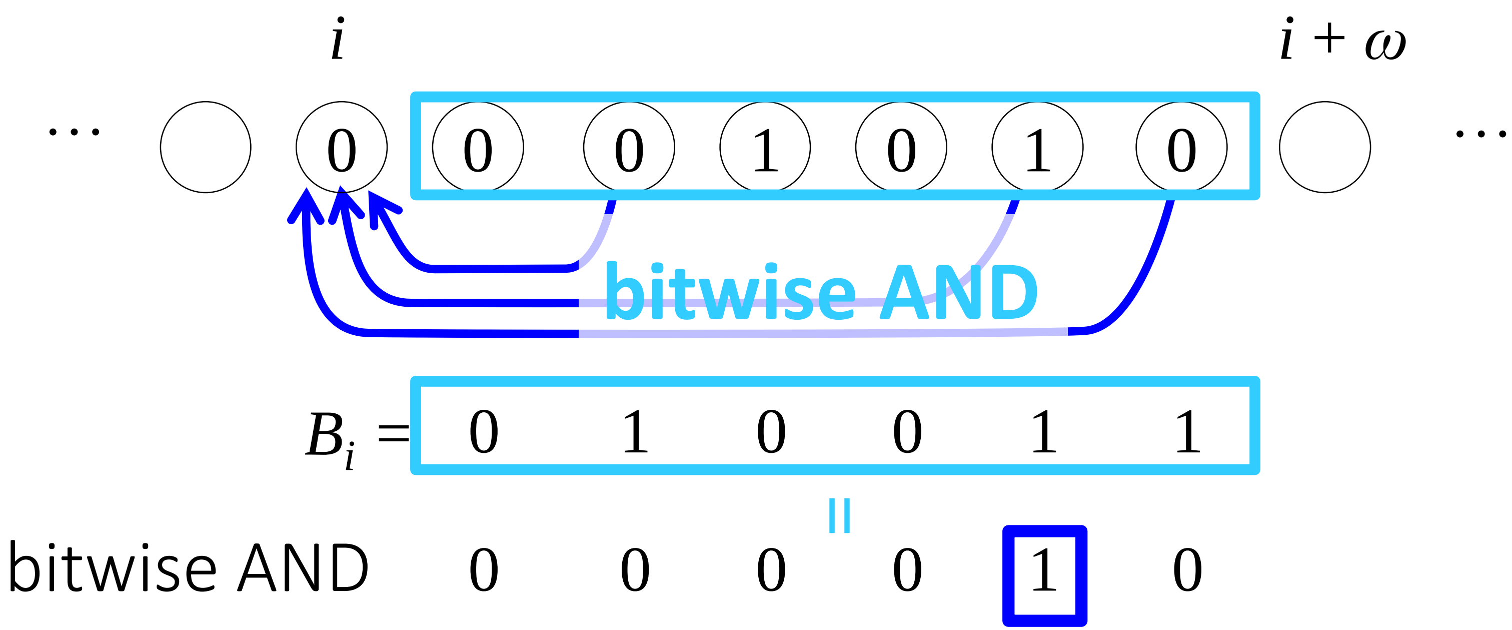
# How to process short edges

- If there is a 1 in the resulting bit string, then node $i$ gets a 1.

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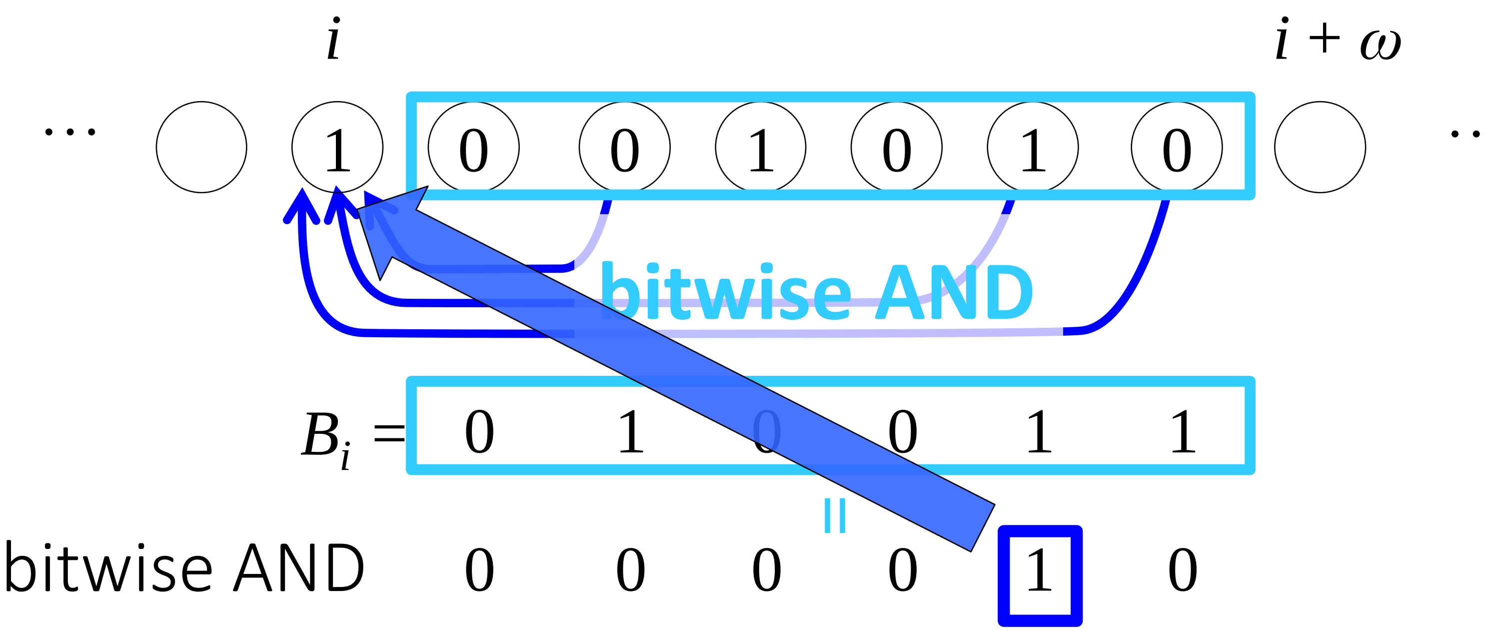


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# Time cost for short edges

- Given bit mask $B_i$, we can process all in-coming short edges of node $i$ in $O(1)$ time.
- An $O(n + \#\text{SPR})$-time preprocessing allows us to compute the bit mask $B_i$ for all nodes $i$.
- Overall, we need $O(n + \#\text{SPR})$ total time for short edges.

# Main result

*Theorem*

Given a string of length $n$, we can compute a square factorization of the string in $O(n)$ time.

- $O(n + \#\mathrm{LPR} + \#\mathrm{SPR} + (n \log n)/\omega)$ time.
  - $\#\mathrm{LPR} + \#\mathrm{SPR} < n$ [Bannai et al., 2015]
  - $(n \log n)/\omega = O(n)$ because $\omega = \Omega(\log n)$.
- Hence, it takes $O(n)$ total time.

# Open questions

- Is it possible to compute a square factorization in $O(n)$ time *without bit parallelism*?
- Is it possible to compute largest/smallest square factorizations in $O(n)$ time?

- It is possible to compute largest/smallest *repetition factorization* in $O(n \log n)$ time [PSC 2016, accepted].
  - Here each factor is a *repetition* of form $x^k x'$ with $k \geq 2$ and $x'$ being a prefix of $x$.
  - $O(n)$-time algorithm exists for this?