Summer School @ U. Helsinki, 2016

The myriad virtues of DAWGs

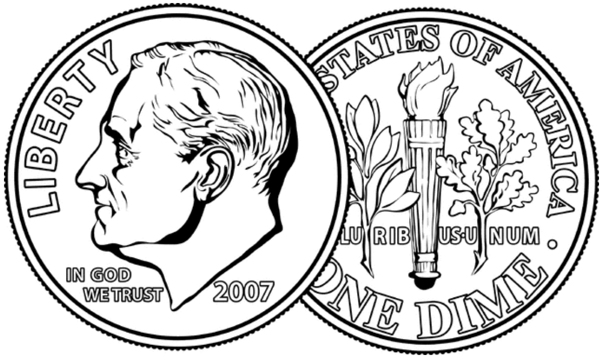
Shunsuke Inenaga
Kyushu University, Japan

Overview of This Course

- Suffix tree [Weiner, 1973] is a fundamental data structure for string processing.
- "The myriad virtues of subword trees" [Apostolico, 1985]
- Directed acyclic word graph (DAWG) [Blumer et al., 1985] is a "dual" data structure for suffix tree.
- In this course, we study some nice properties and usefulness of DAWGs.

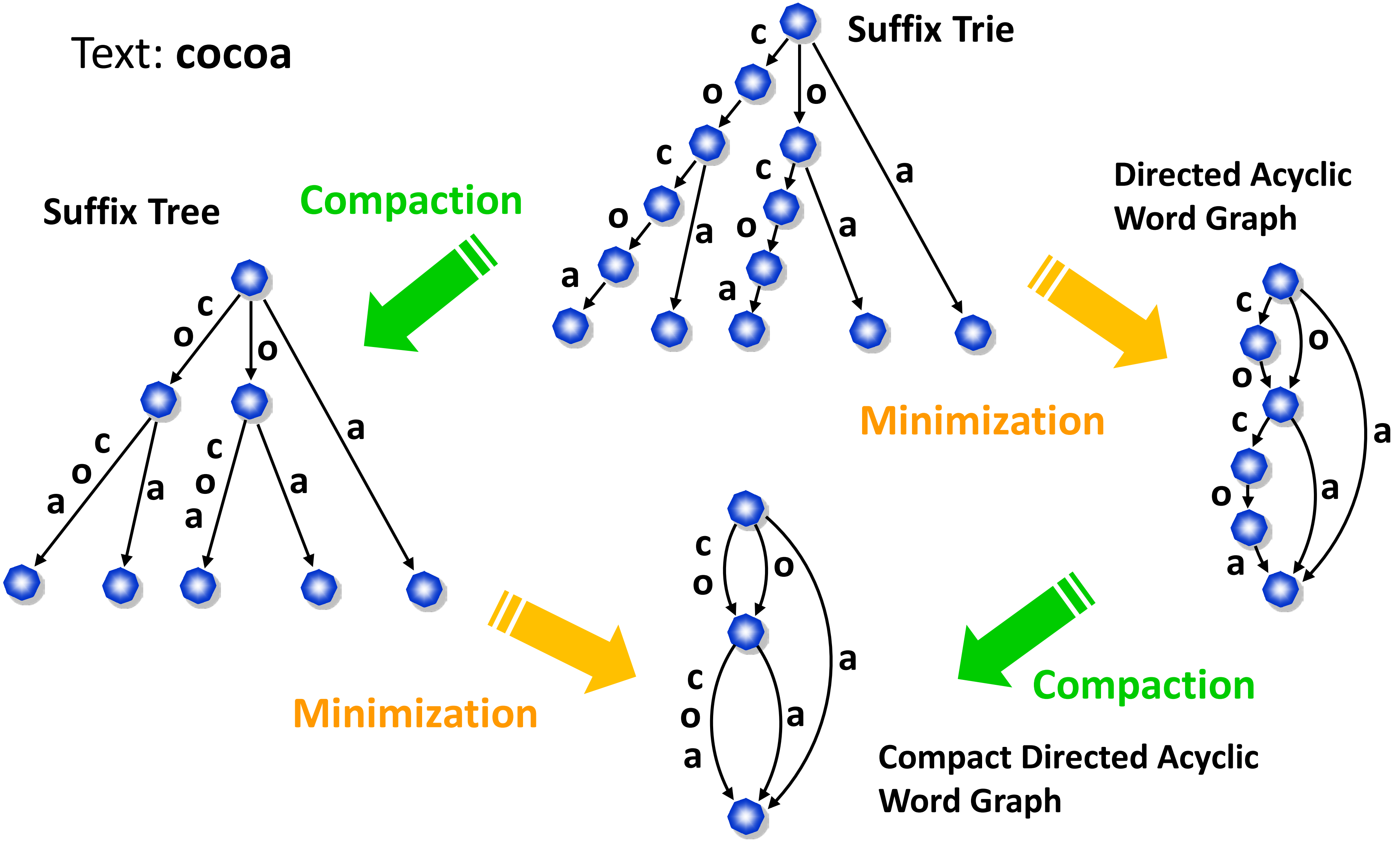
Relation to Previous Course

- This course will share some consequences with the previous course by Djamal & Fabio.
- This happens due to the “duality” between DAWGs and suffix trees, i.e., we will see “the same coin from both sides”.



Text Indexing Structures

Text: **cocoa**



Suffix Trie

Suffix Tree

Directed Acyclic Word Graph

Compact Directed Acyclic Word Graph

Compaction

Minimization

Minimization

Compaction

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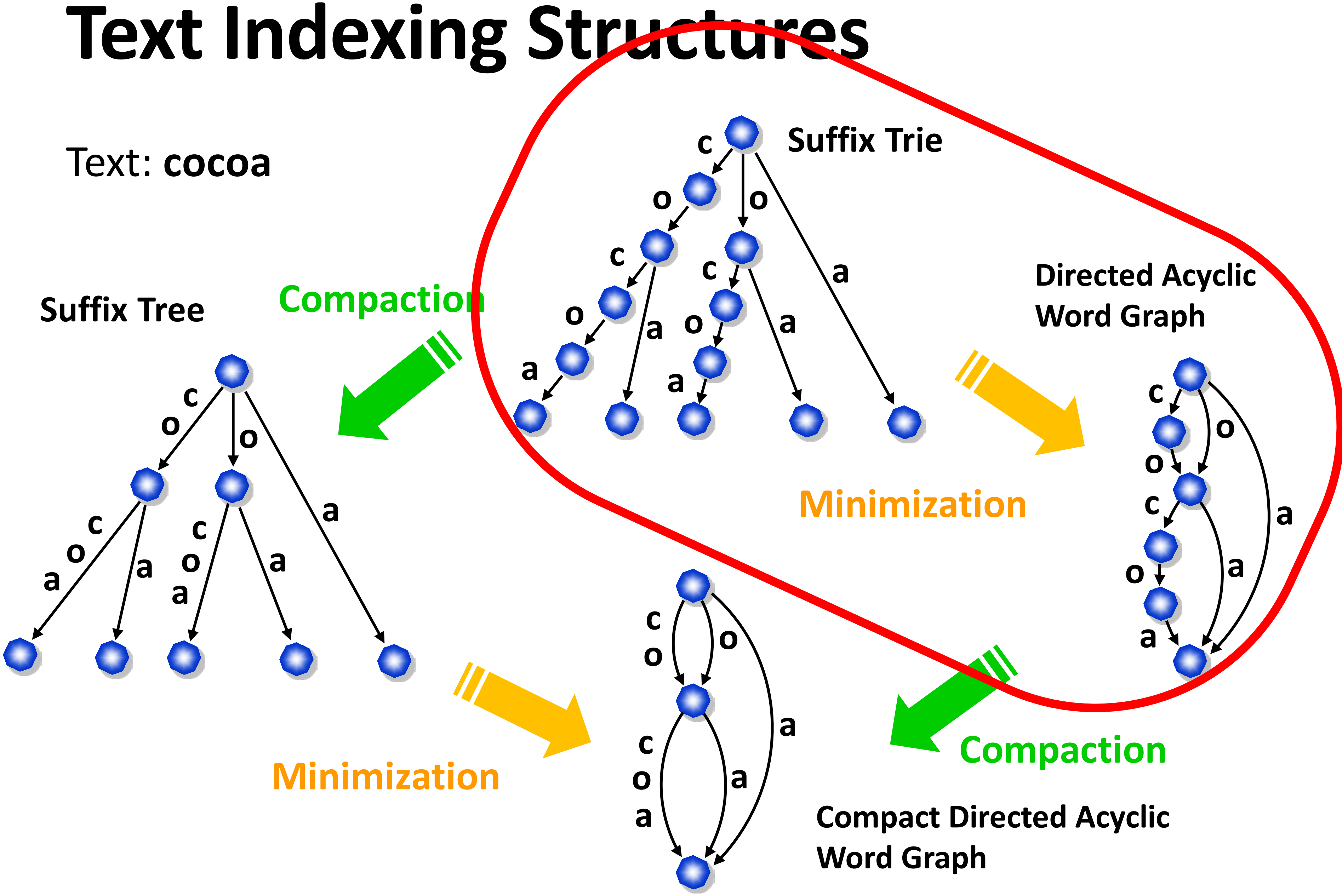

Suffix Trie

Suffix Tree

Directed Acyclic Word Graph

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Minimization

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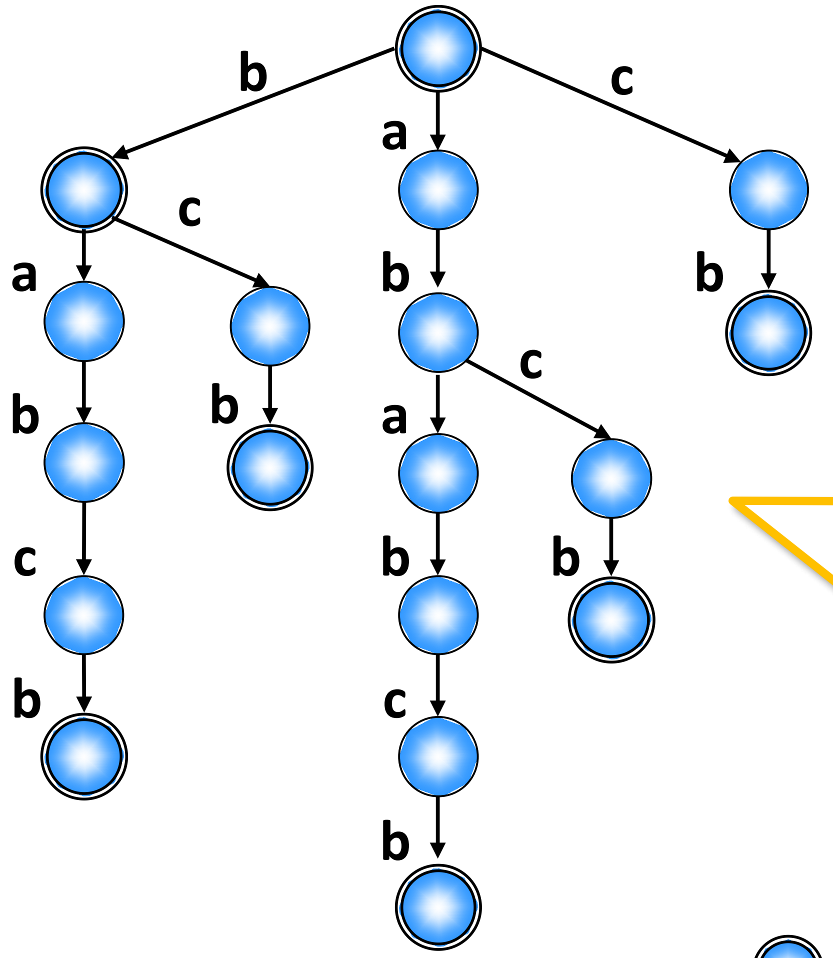
From Suffix Trie to DAWG

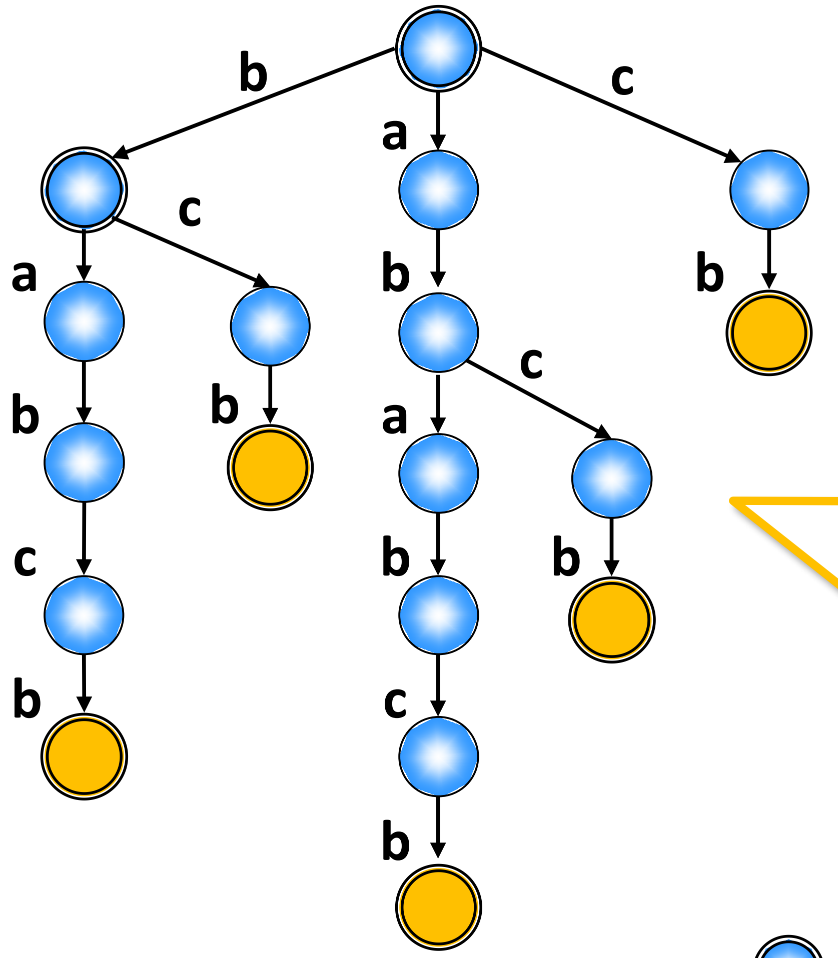


ababcb

Merge isomorphic subtrees (subgraphs) bottom up.

● accepting nodes for suffixes

From Suffix Trie to DAWG



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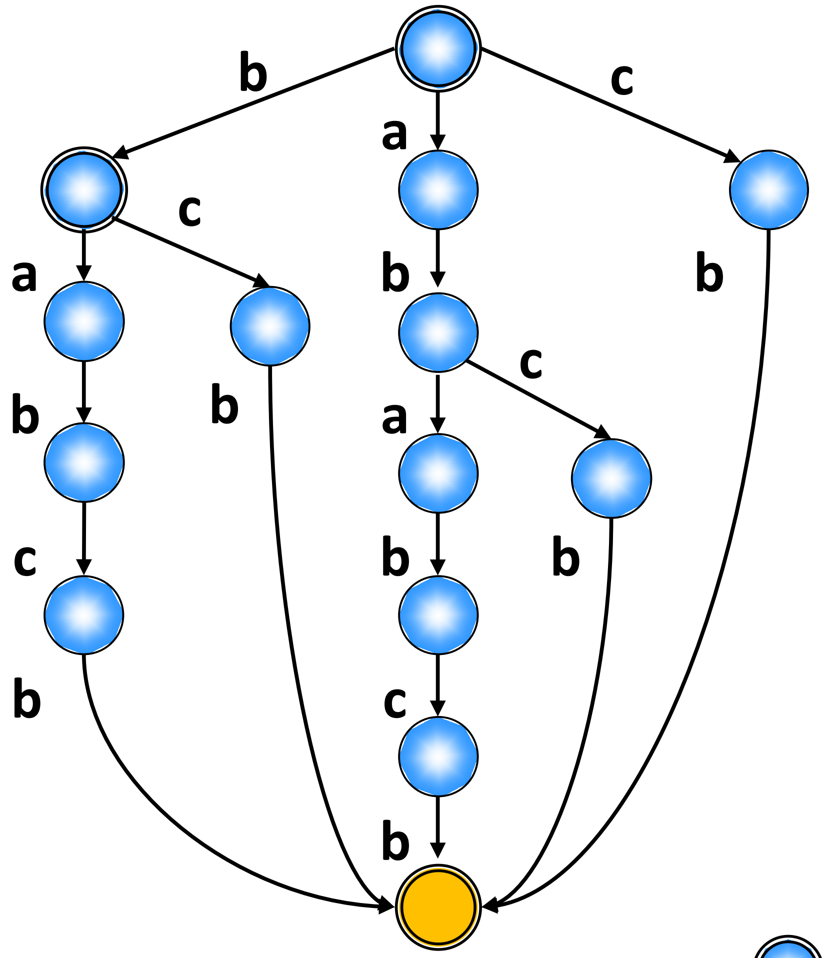

ababcb

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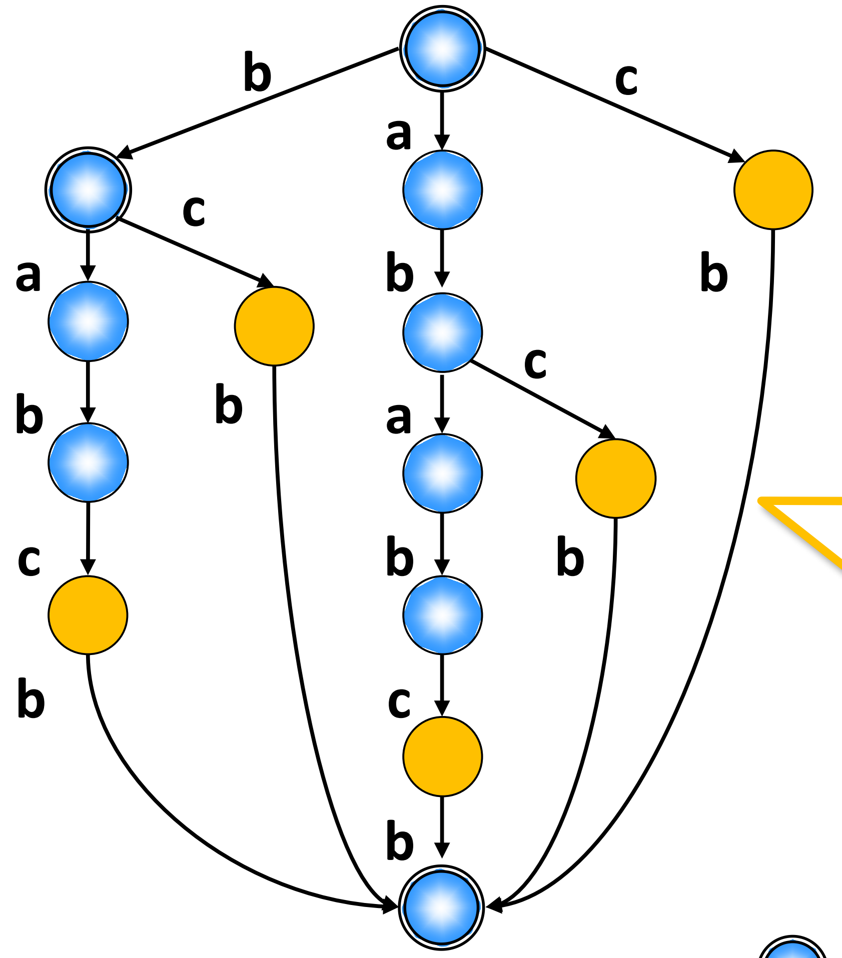

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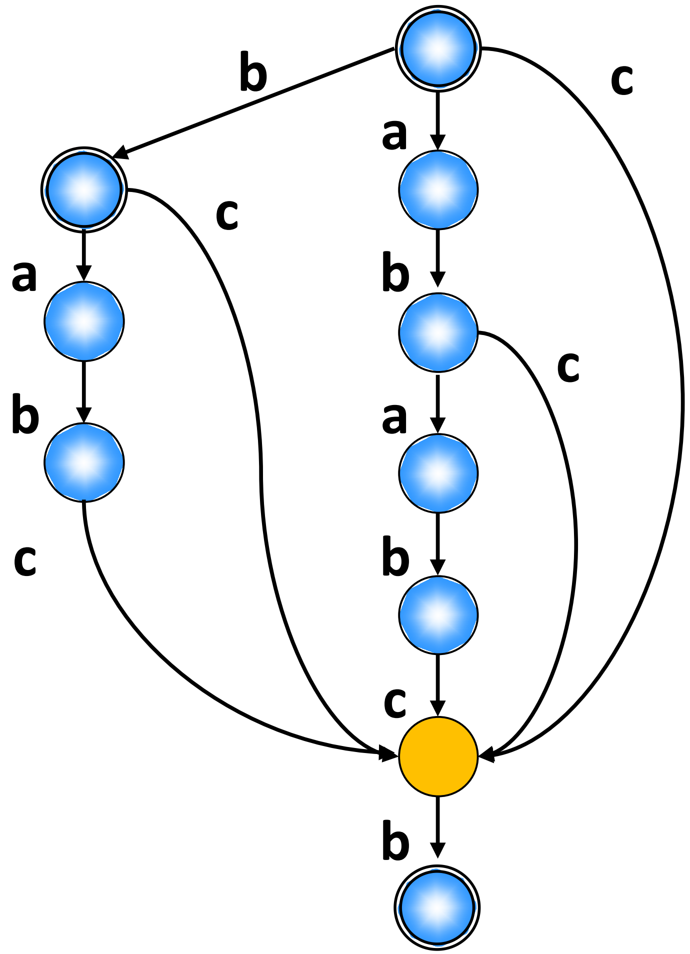

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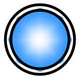
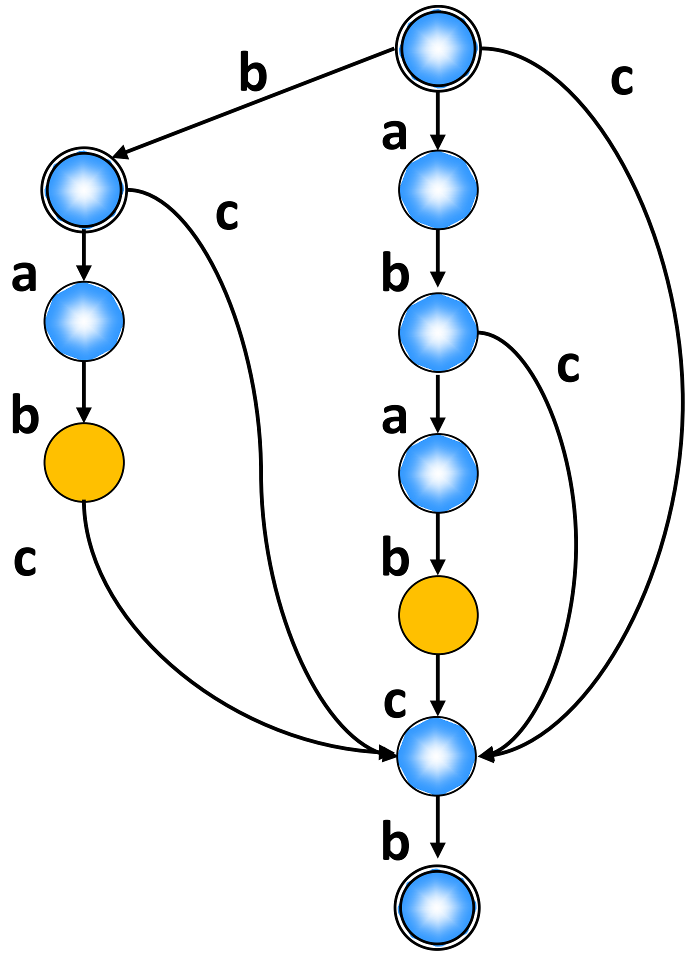


From Suffix Trie to DAWG



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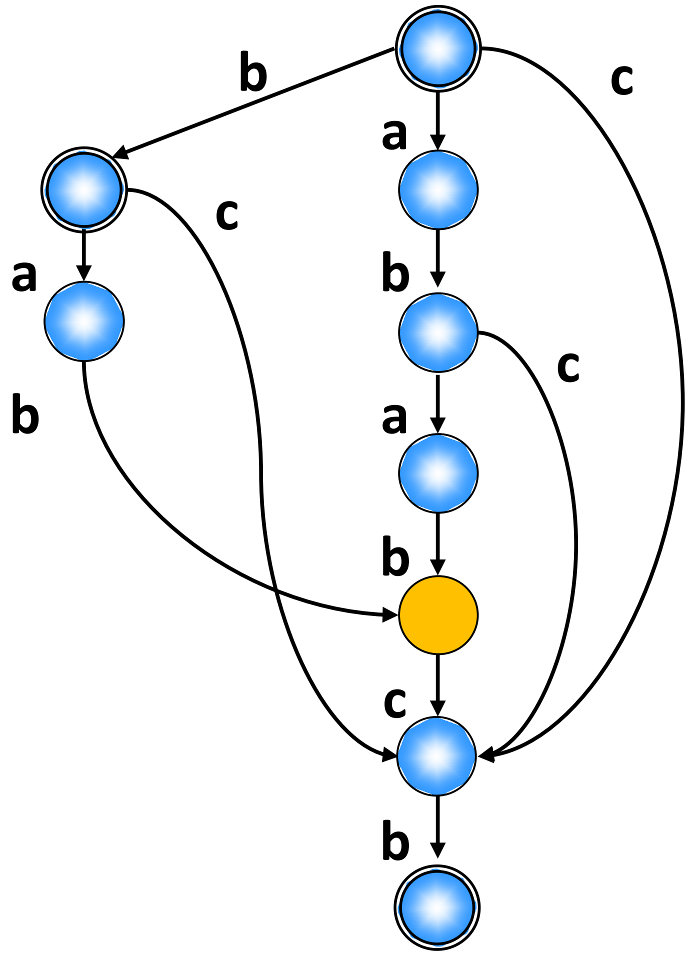
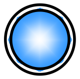
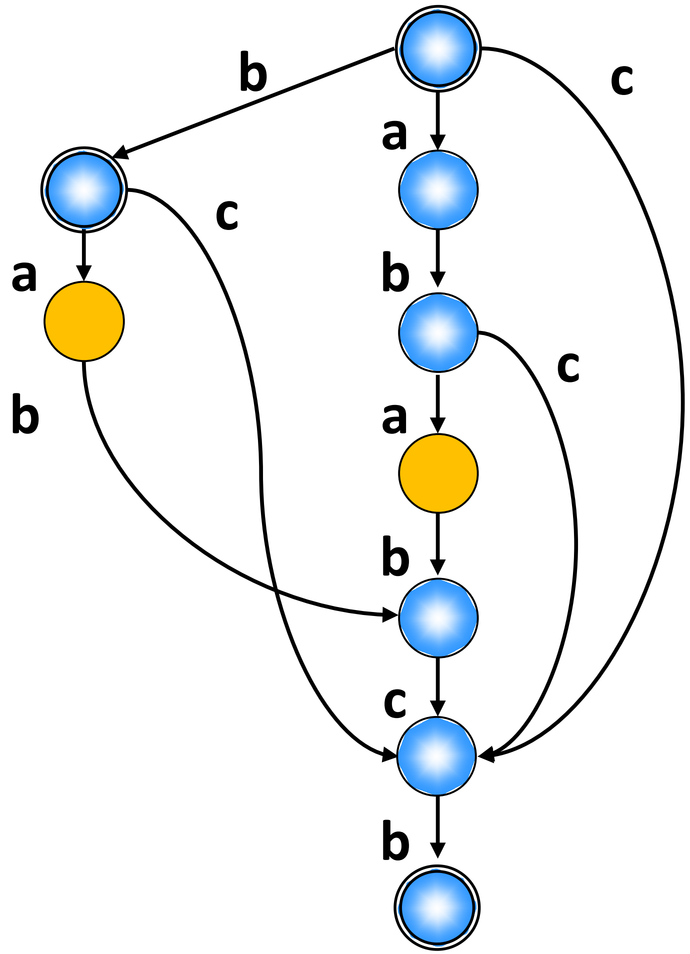


ababcb

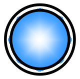


From Suffix Trie to DAWG

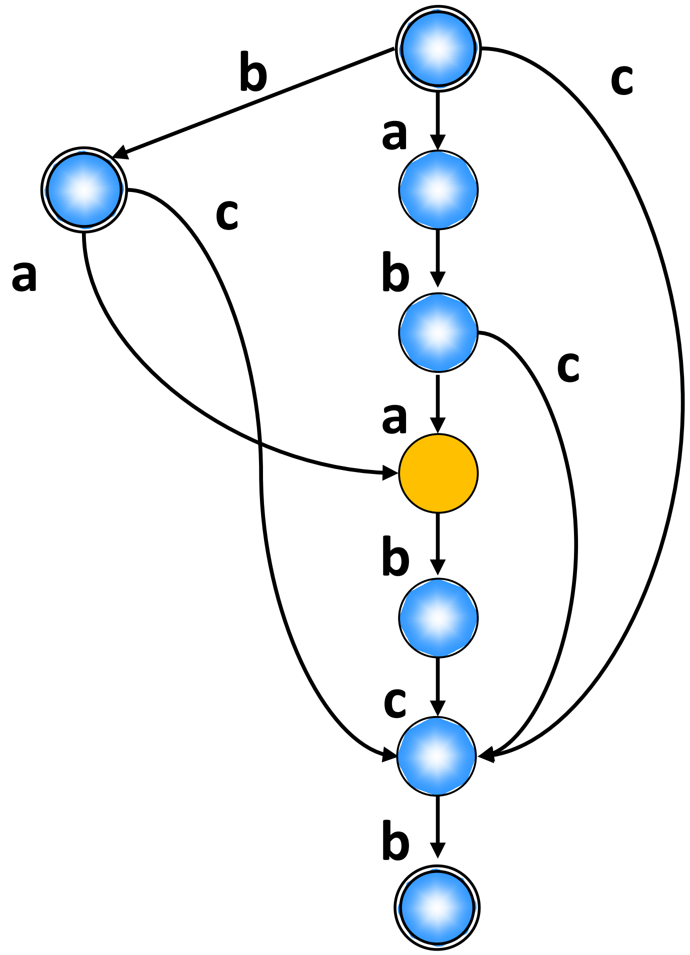


ababcb

Merge isomorphic subtrees (subgraphs) bottom up.


accepting nodes for suffixes

From Suffix Trie to DAWG



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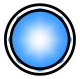
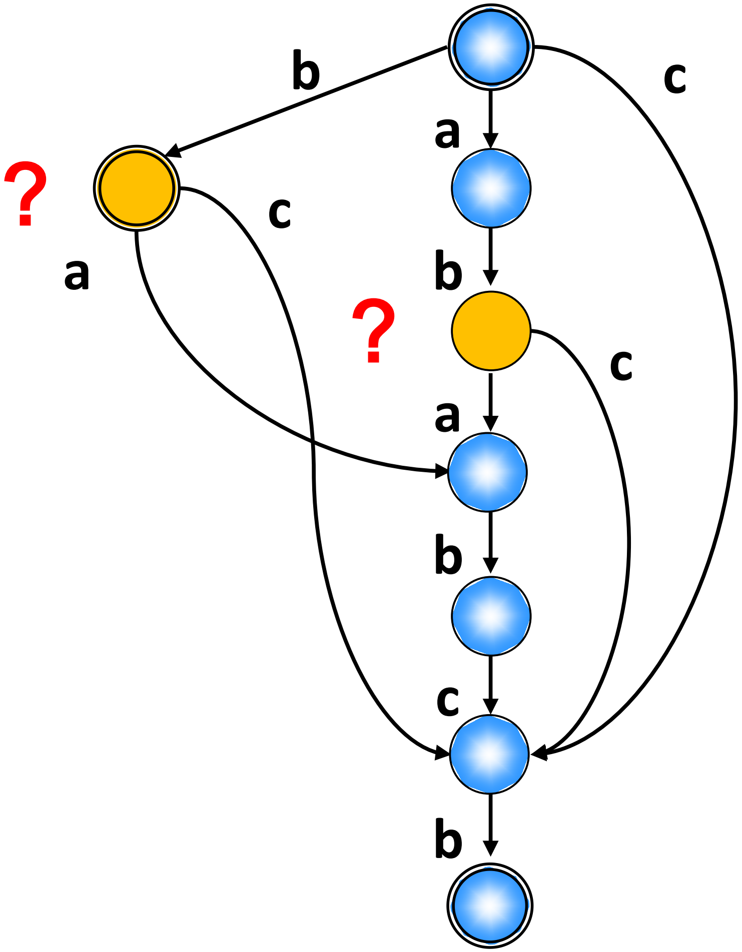
 accepting nodes for suffixes

From Suffix Trie to DAWG

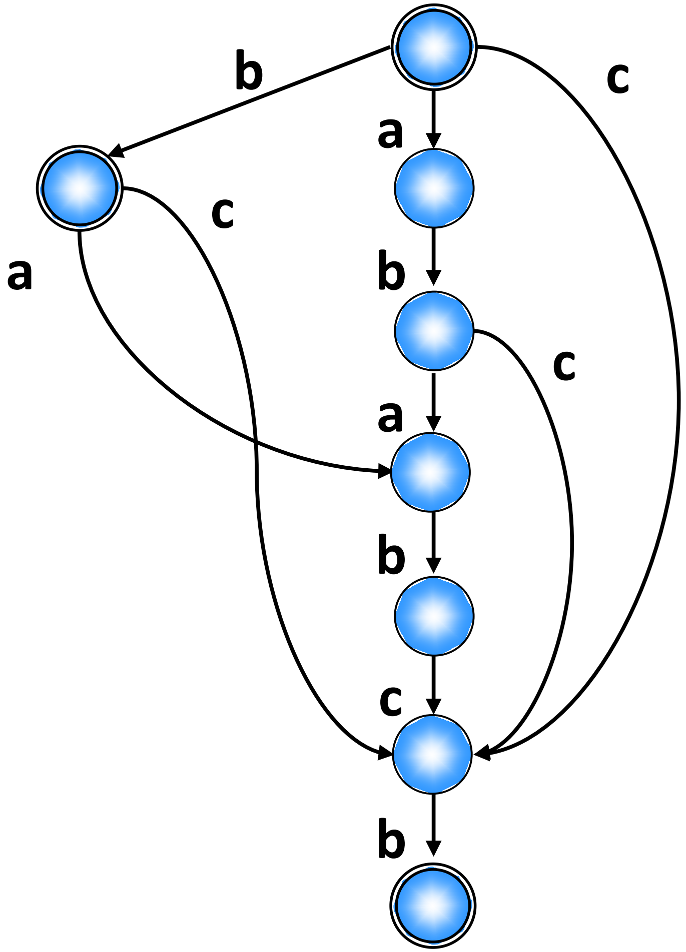


ababcb

One is accepting state, while the other isn't. Don't merge them.

● accepting nodes for suffixes

From Suffix Trie to DAWG



ababcb

DAWG of string
ababcb

◎ accepting nodes for suffixes

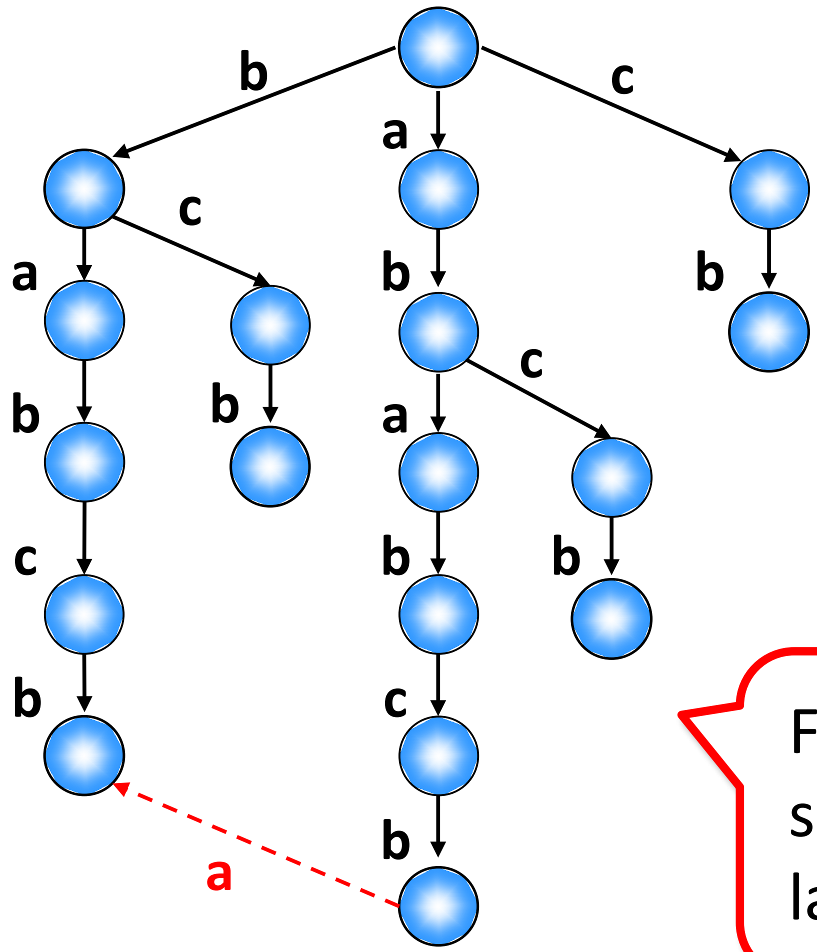
Minimality of DAWG

Theorem 1

The DAWG of string w is the **smallest automaton** that recognizes all suffixes of w.

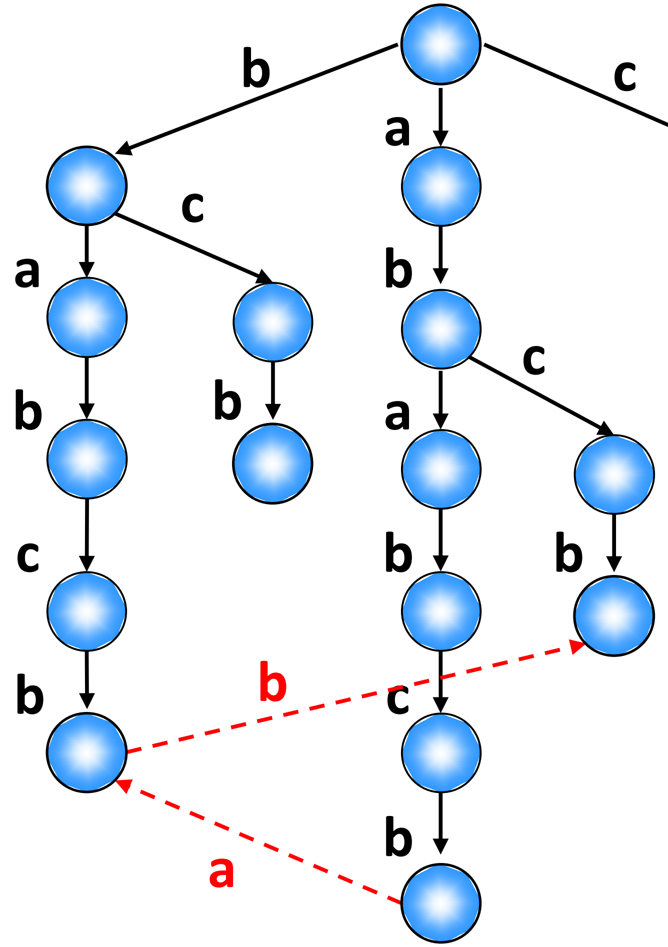
- Clearly, the suffix trie of w recognizes all suffixes of w, so does the DAWG of w.
- By construction, the DAWG is the smallest such automaton.

Suffix Links of Suffix Trie



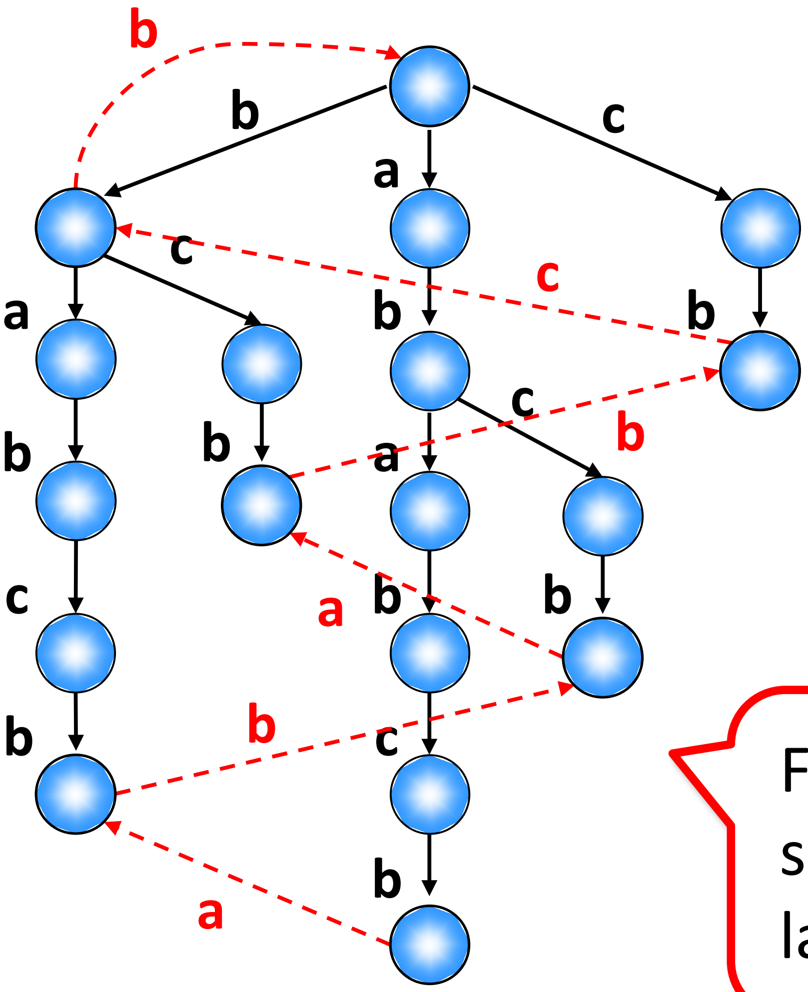
For letter ***a*** and string ***x***, suffix link from node ***ax*** is labeled ***a*** and goes to node ***x***.

Suffix Links of Suffix Trie



For letter a and string x, suffix link from node ax is labeled a and goes to node x.

Suffix Links of Suffix Trie



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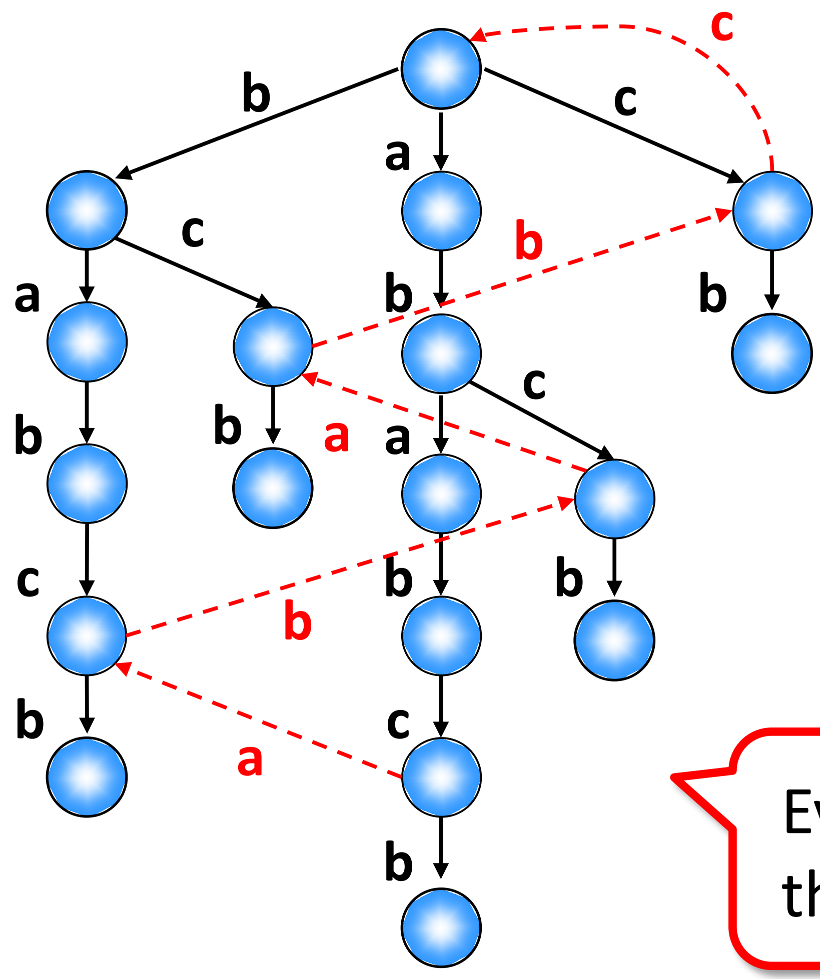
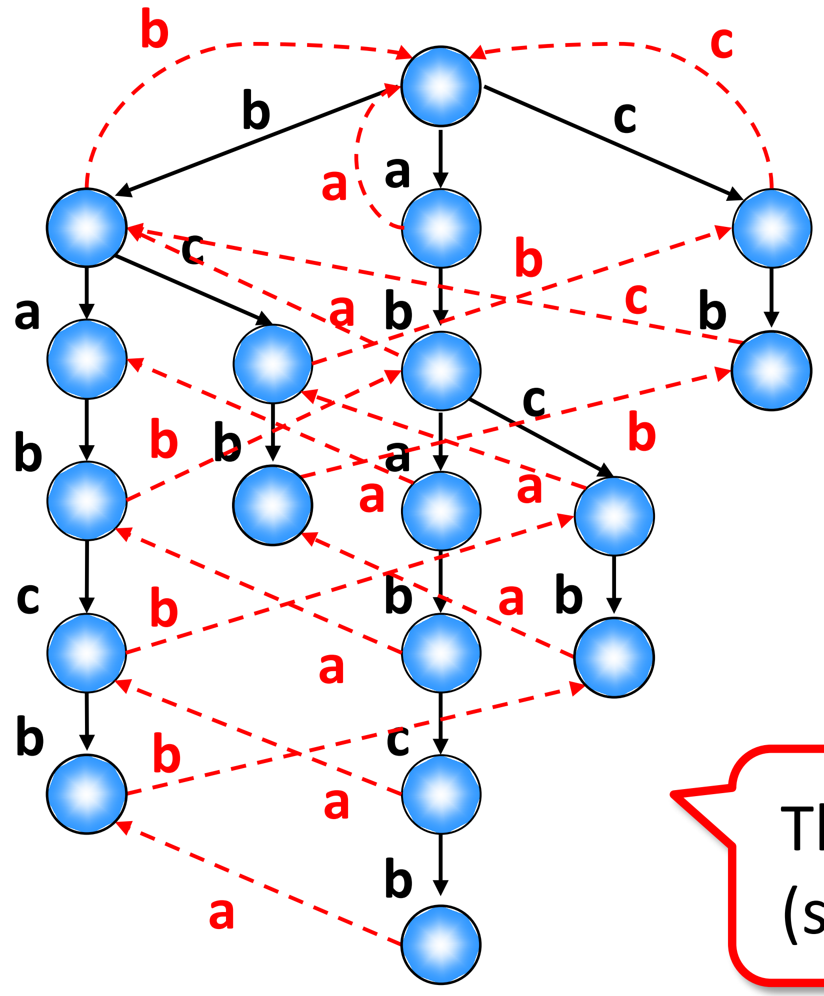


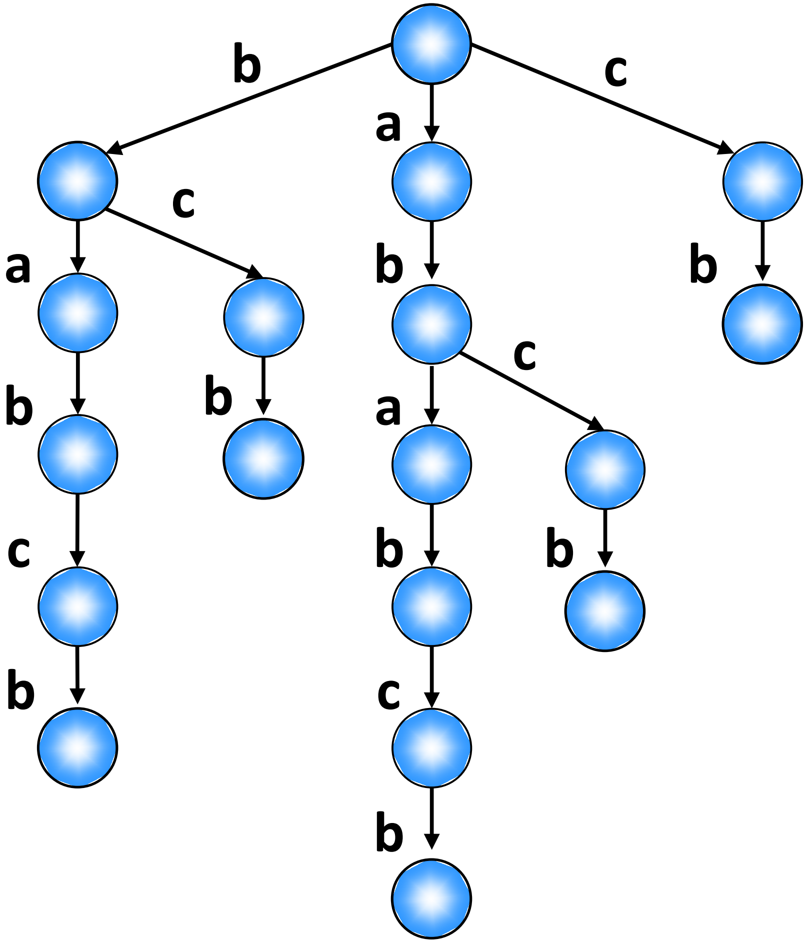
Every suffix link path reaches the root.

Suffix Link Tree of Suffix Trie

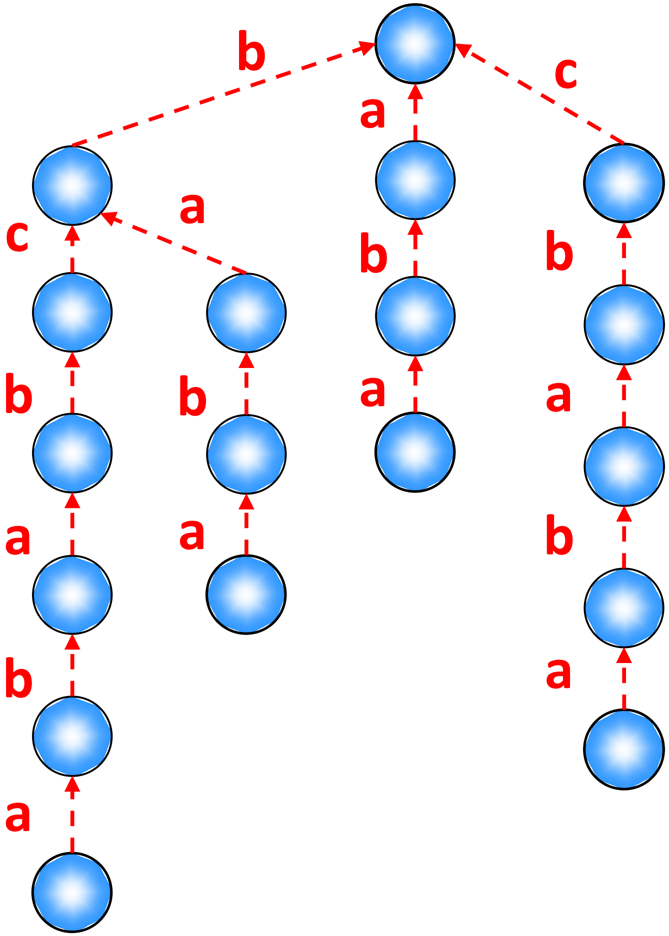


The suffix links form a tree (suffix link tree).

Suffix Link Tree of Suffix Trie

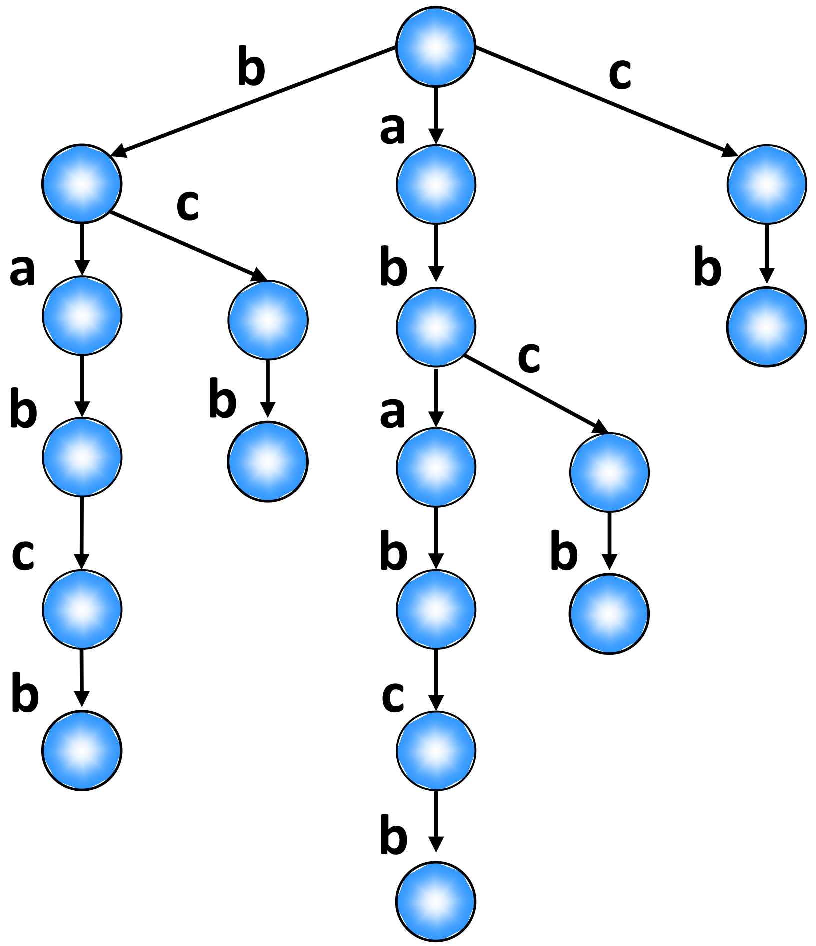


Suffix Trie of **ababcb**

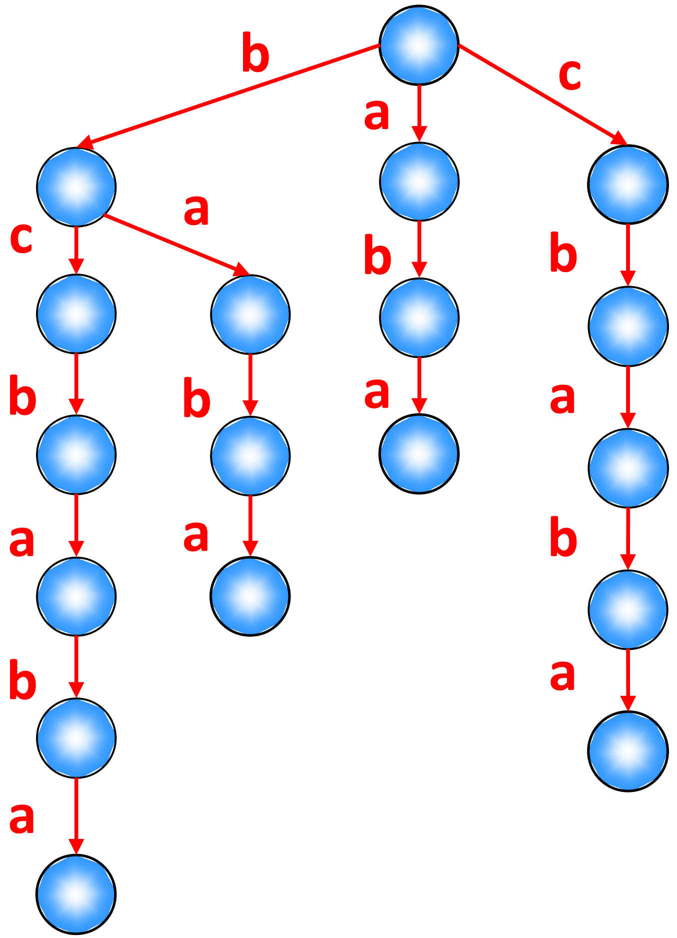


Suffix Link Tree of **ababcb**

Suffix Link Tree = Suffix Trie of reverse



Suffix Trie of **ababcb**



Suffix Trie of **bcbaba**

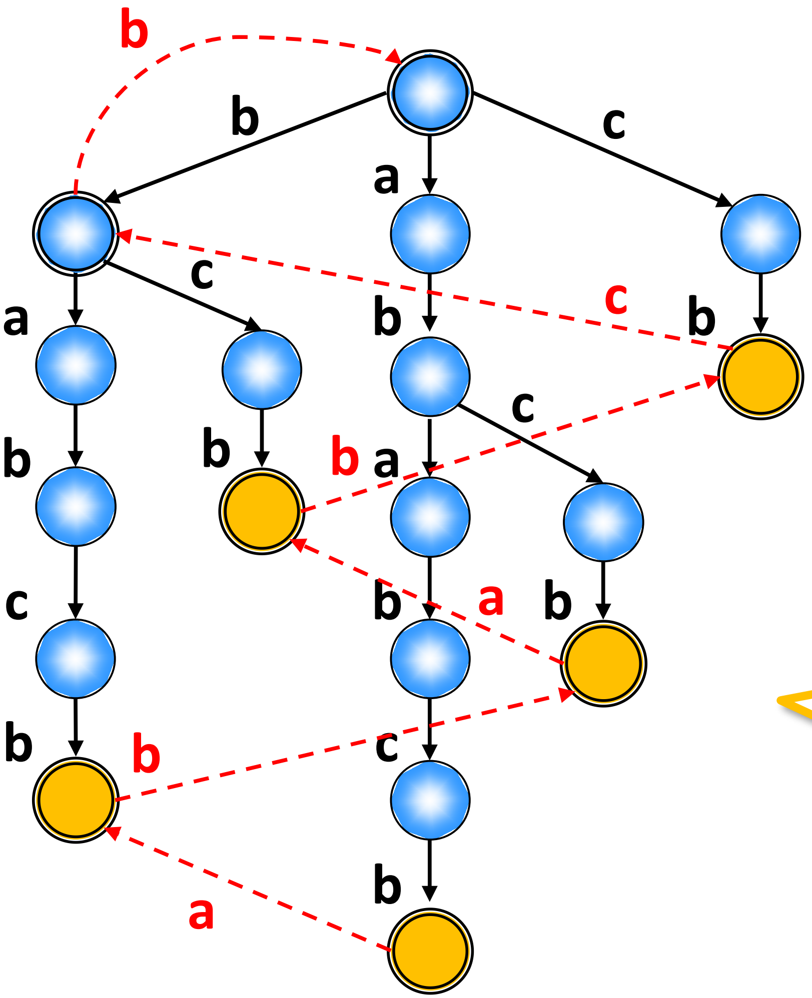
Suffix Link Tree = Suffix Trie of reverse

Lemma 1

The suffix link tree of the suffix trie of string w forms the suffix trie of reversed string w^R.

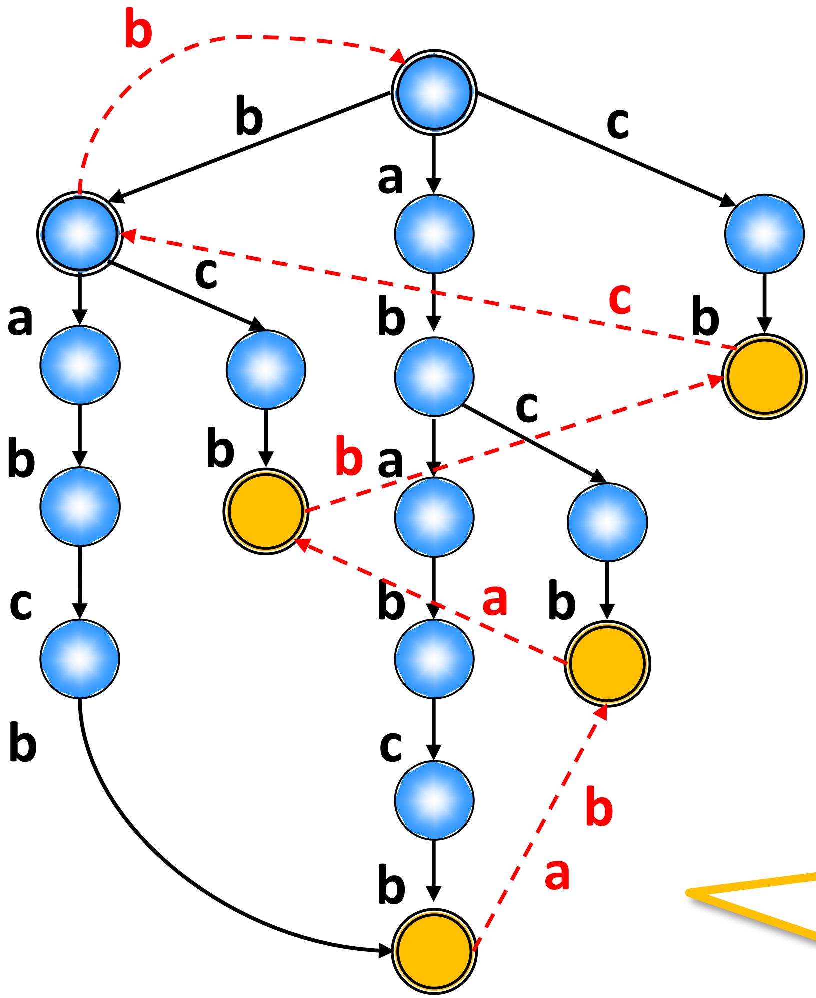
- For any node of the suffix trie of w which represents substring x of w, the node represents subtring x^R of w^R in the suffix link tree.

Node Merges and Suffix Links



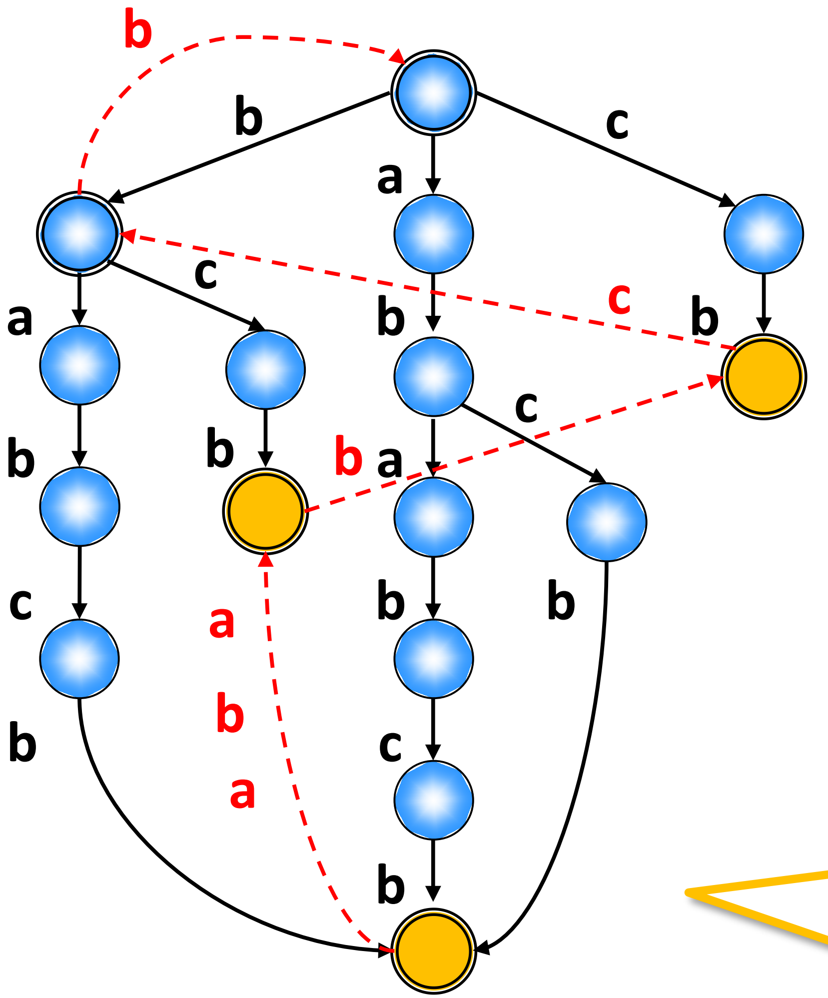
Merged nodes are connected by a chain of suffix links.

Node Merges and Suffix Links



When merging nodes, we also contract the suffix links.

Node Merges and Suffix Links



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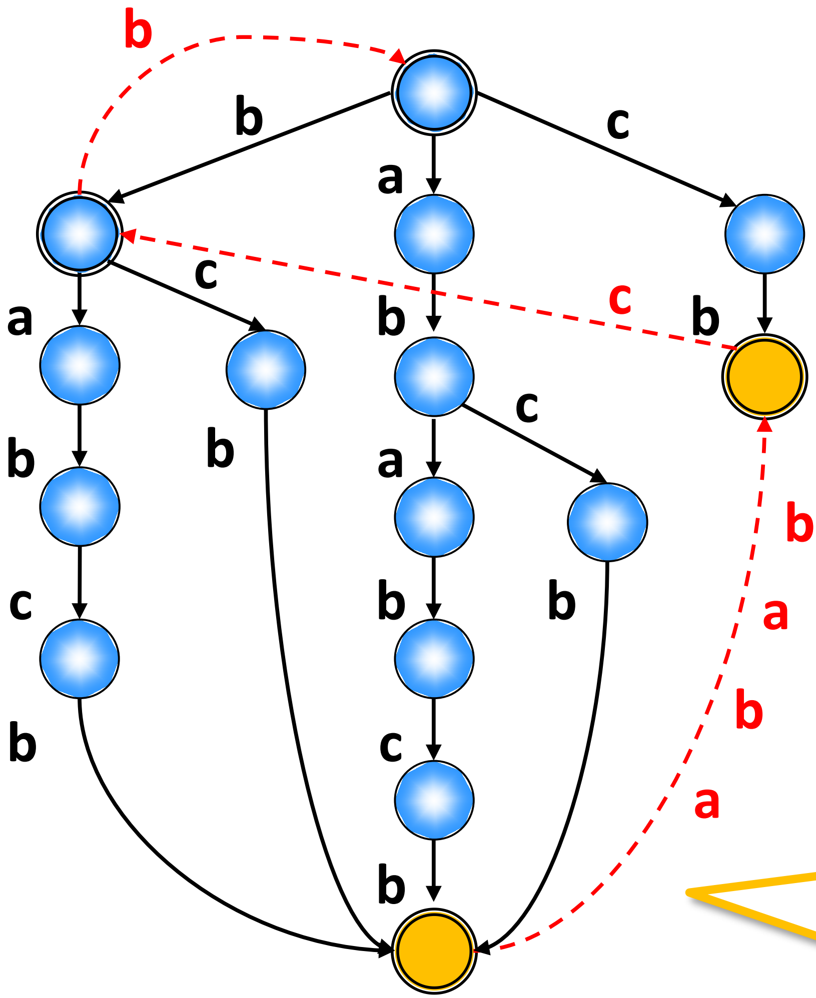

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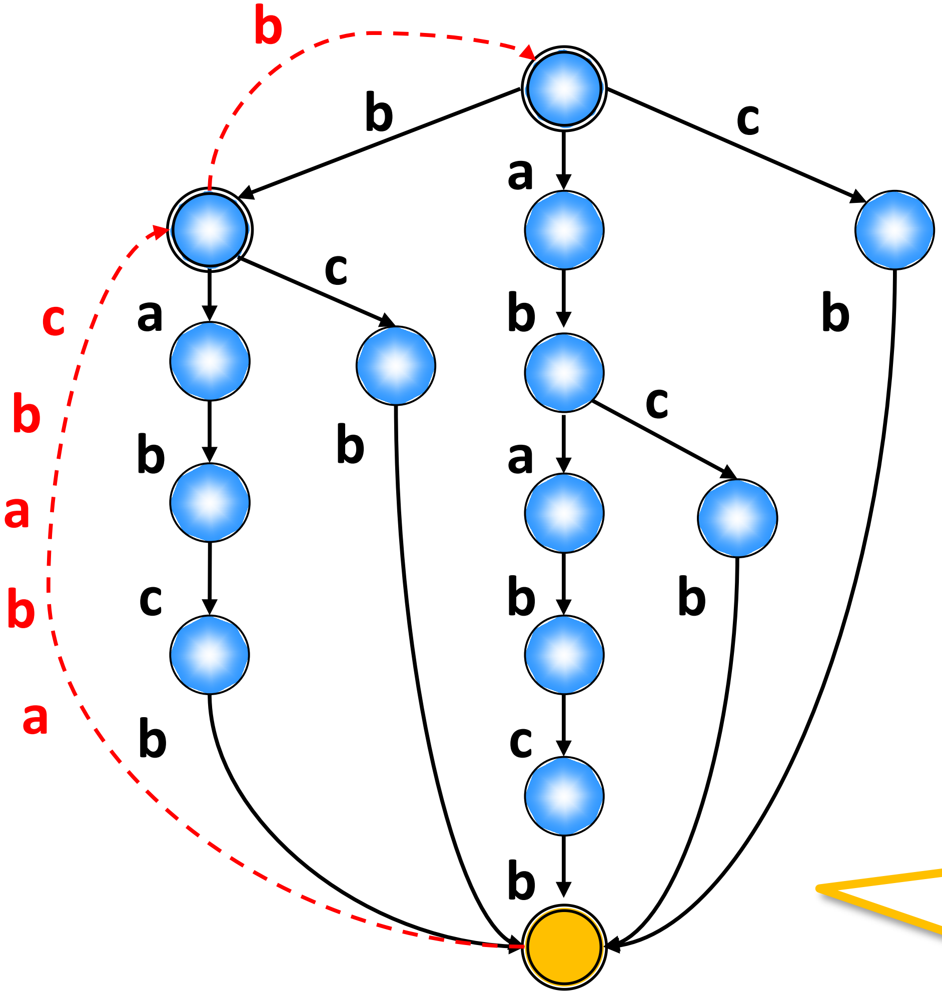

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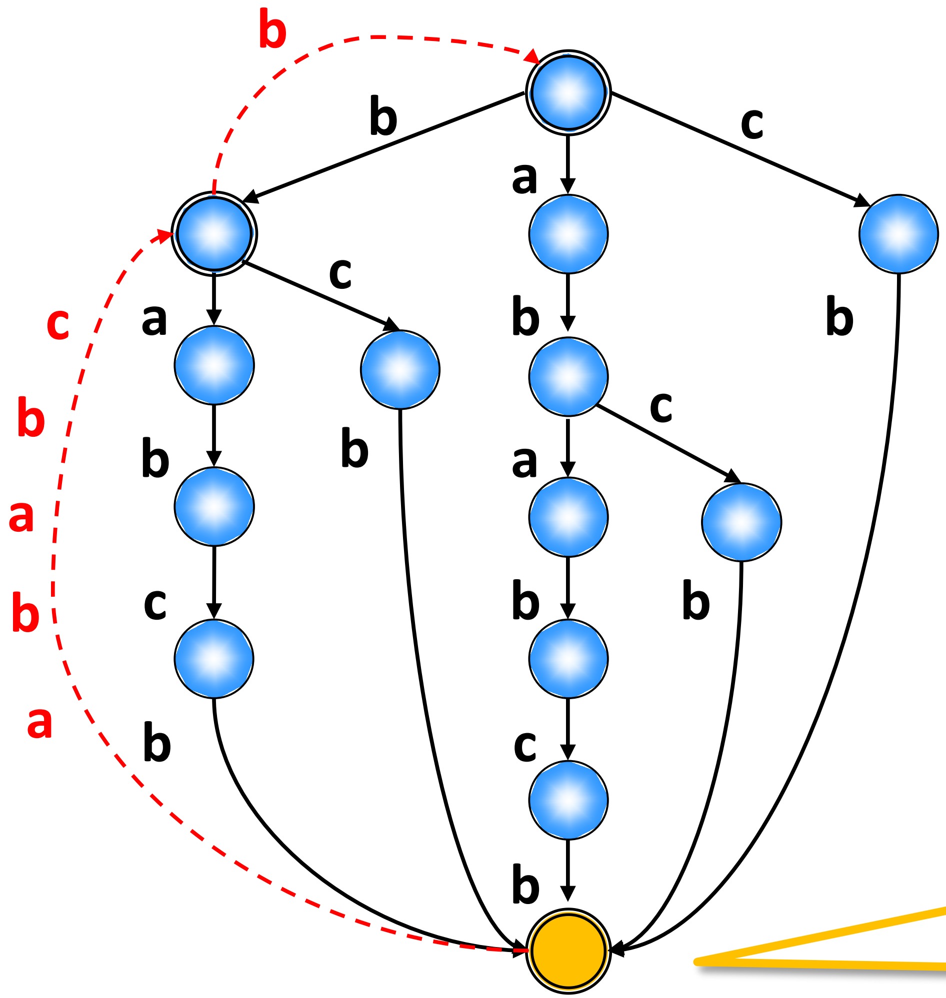
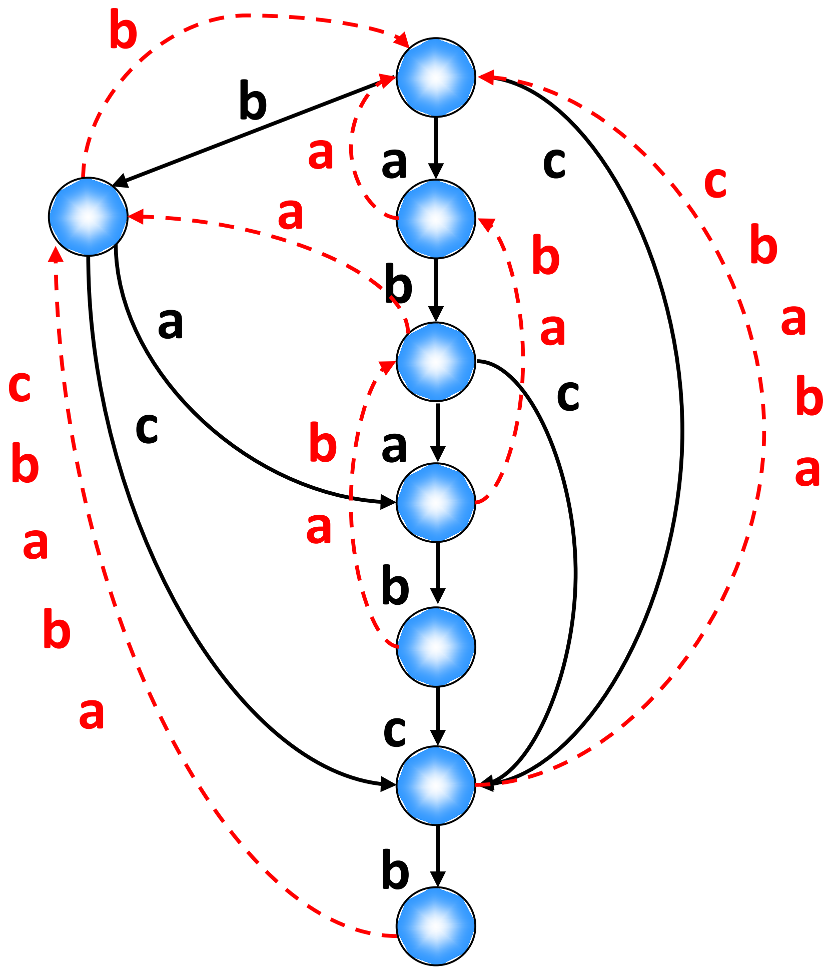
Node Merges and Suffix Links



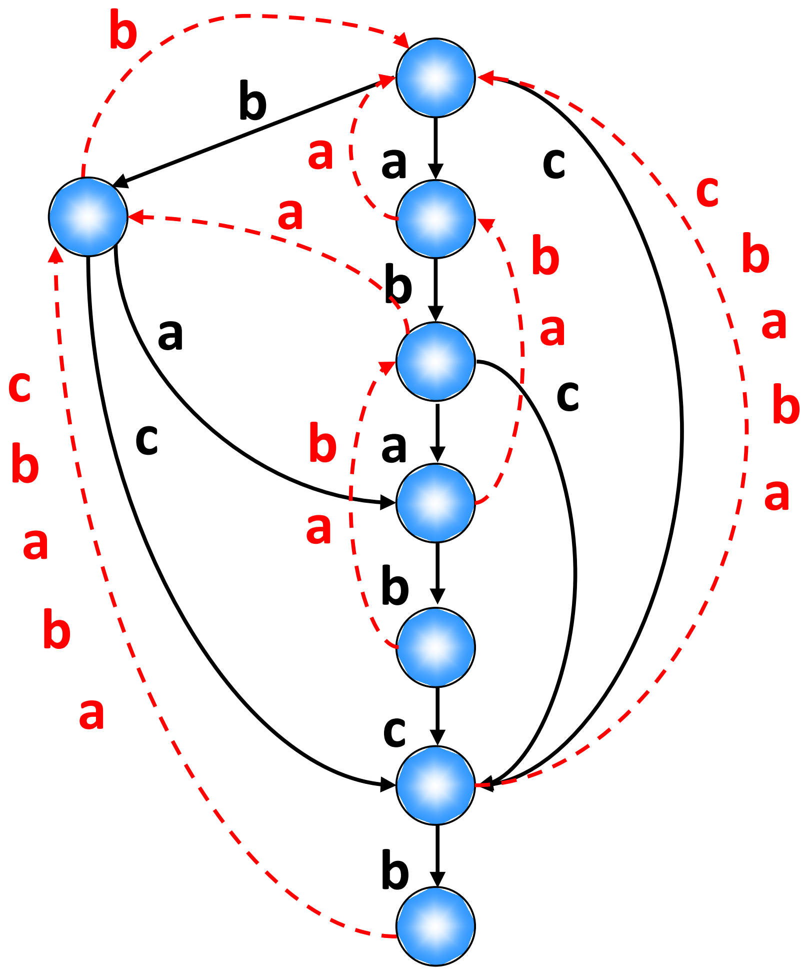
This is the suffix link of this DAWG node.

Suffix Links of DAWG

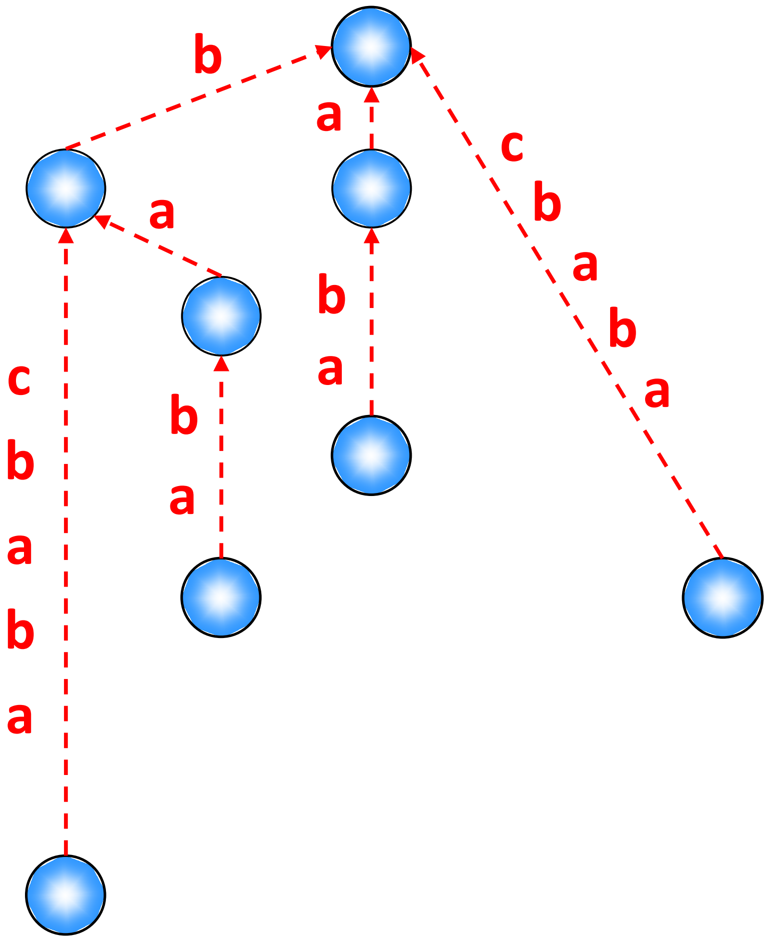


The suffix links of DAWG also forms a tree.

Suffix Links of DAWG

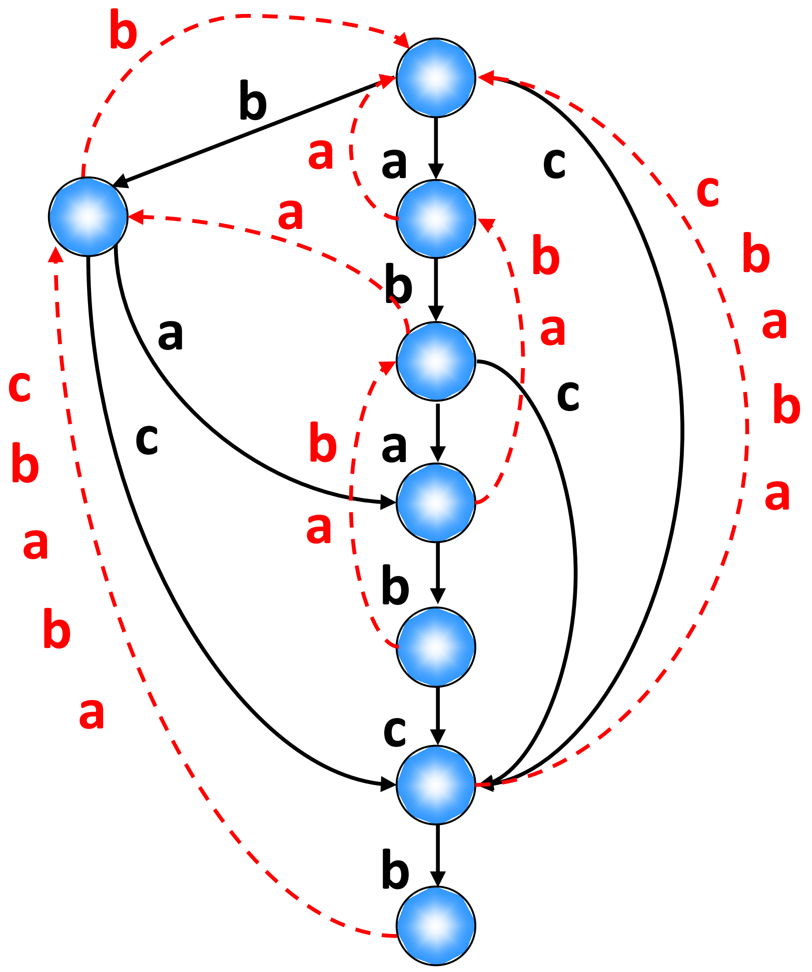


DAWG of **ababcb**

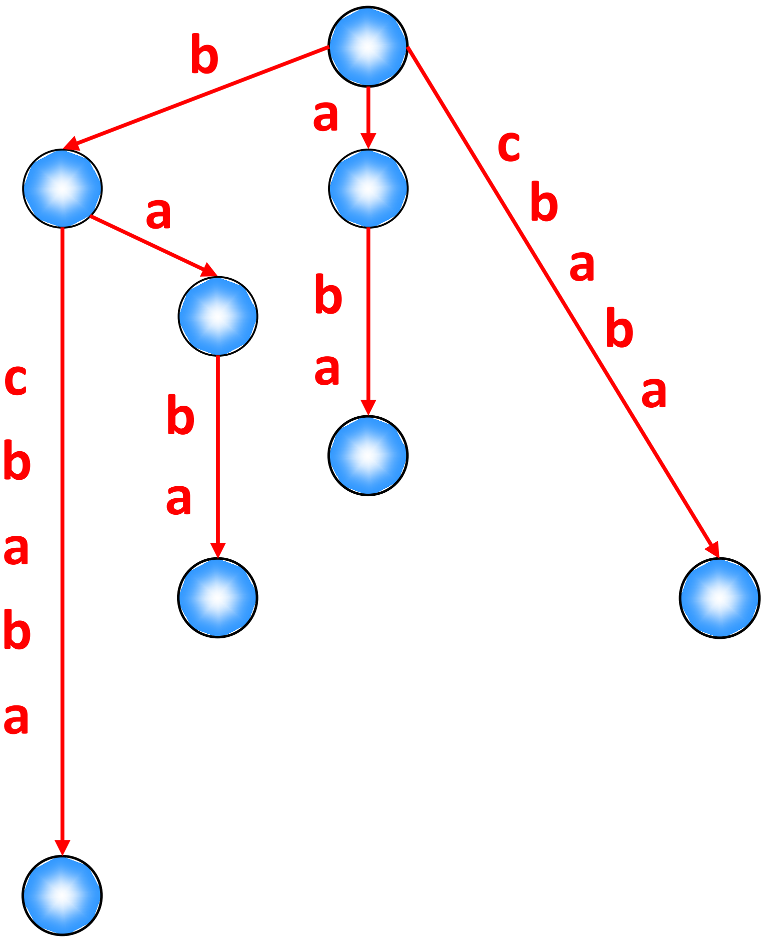


Contracted SLT of **ababcb**

SLT of DAWG = Suffix Tree of reverse

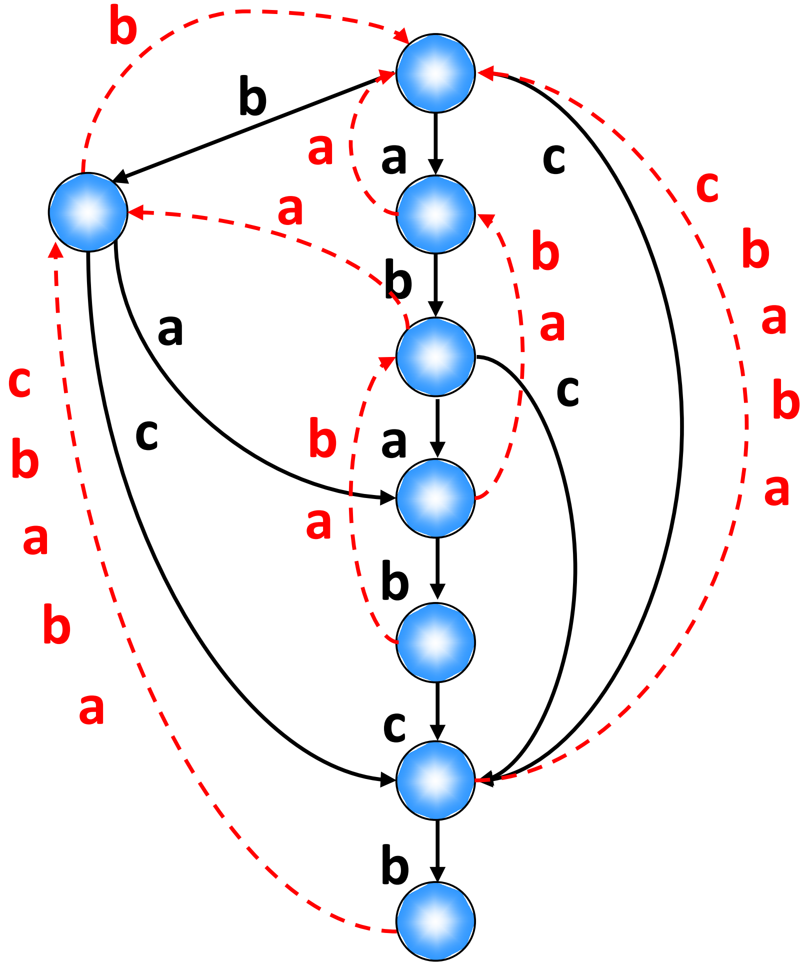


DAWG of **ababcb**

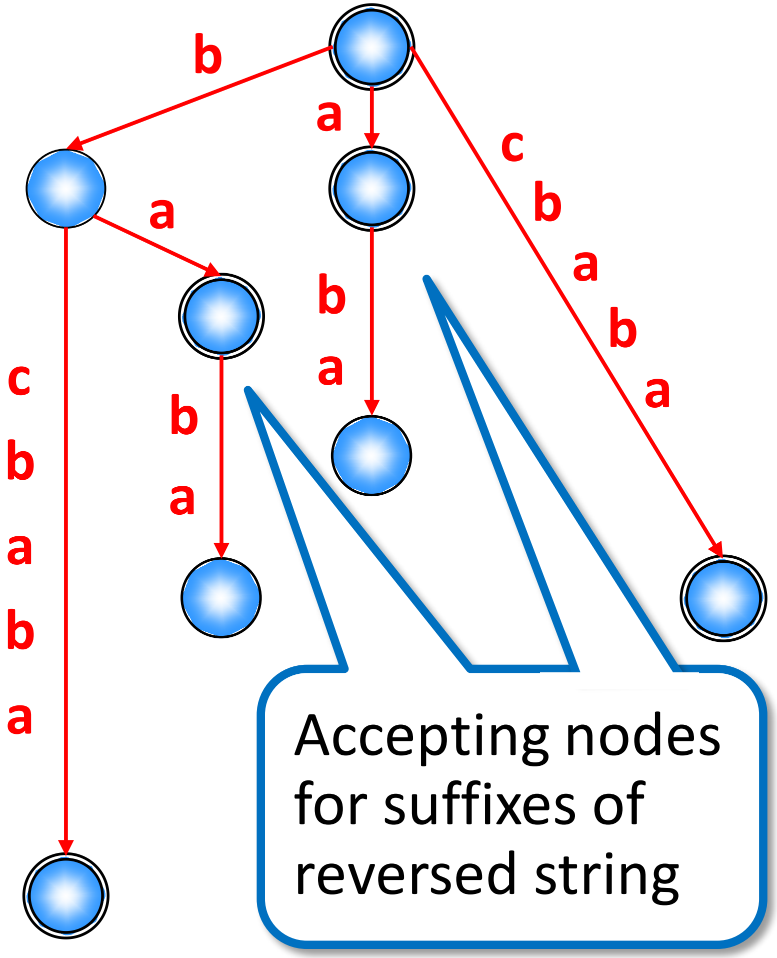


Suffix Tree of **bcbaba**

SLT of DAWG = Suffix Tree of reverse



DAWG of **ababcb**



Accepting nodes for suffixes of reversed string

Suffix Tree of **bcbaba**

SLT of DAWG = Suffix Tree of reverse

Theorem 2

The suffix link tree of the DAWG of string $\boldsymbol{w}$ forms the suffix tree of reversed string $\boldsymbol{w}^{\boldsymbol{R}}$.

- Contracting suffix links during node merges is equivalent to contracting non-branching paths of the suffix trie of $\boldsymbol{w}^{\boldsymbol{R}}$.

Number of Nodes of DAWG

Corollary 1

The number of nodes of the DAWG of any string of length $\boldsymbol{n}$ is at most $\mathbf{2}\boldsymbol{n}\mathbf{-1}$.

- Immediate from Theorem 2.

Number of Edges of DAWG

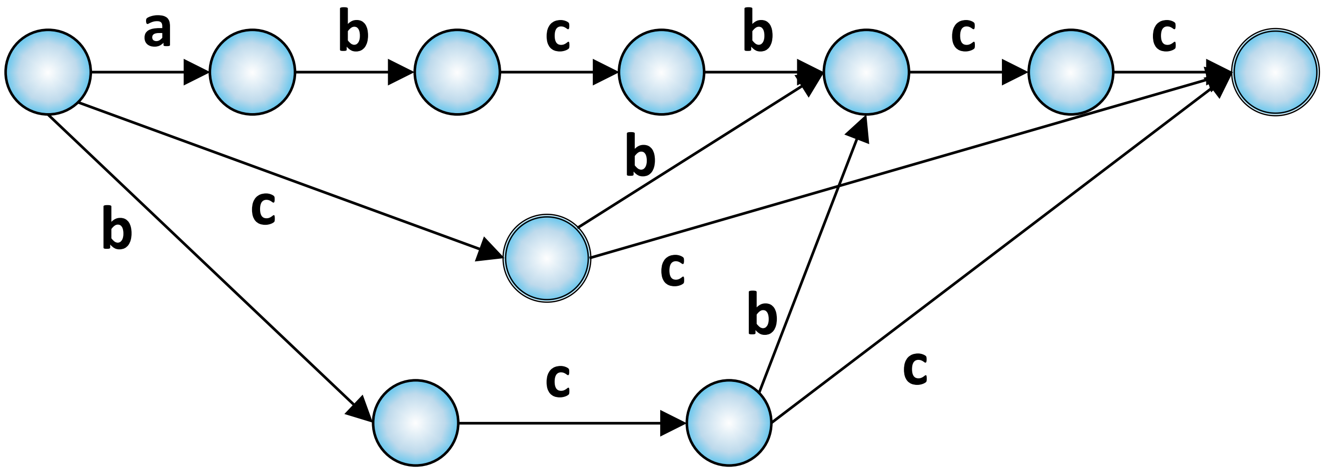
Lemma 2

The number of edges of the DAWG of any string of length n is at most $3n$-3.

Number of Edges of DAWG

Proof.

- Consider any spanning tree T of the DAWG.



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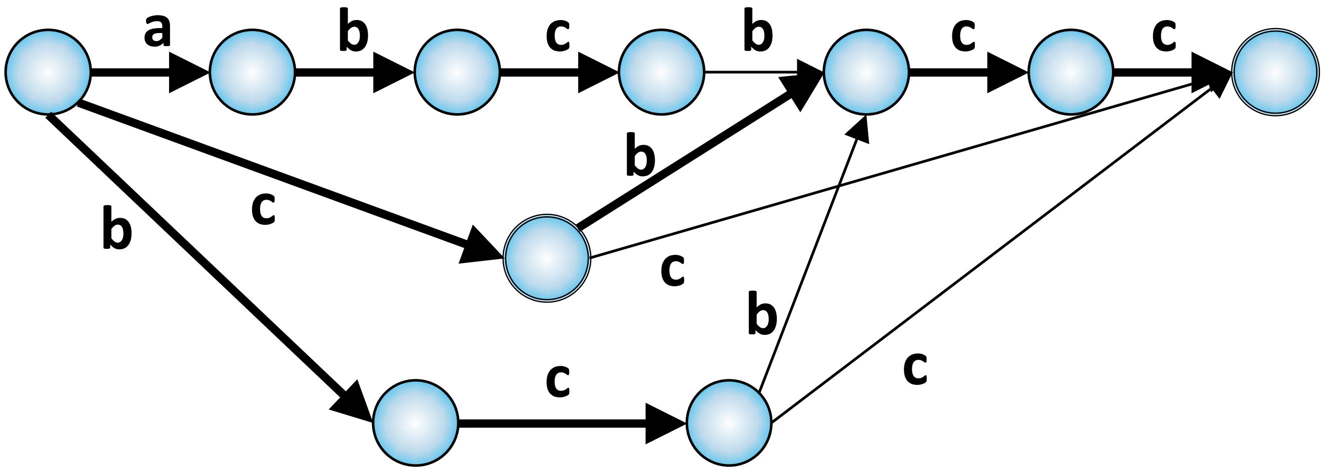
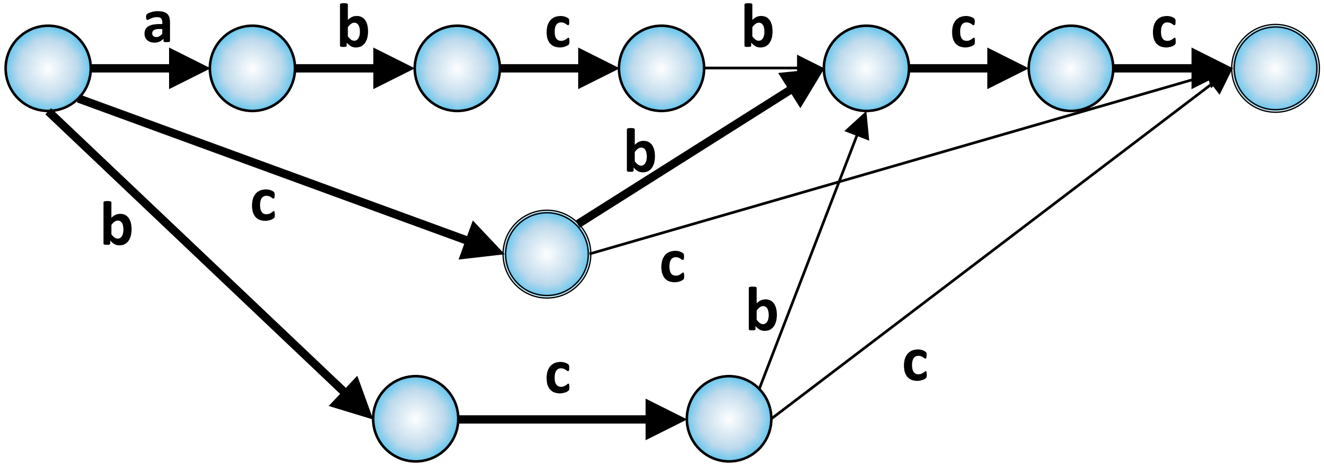


Number of Edges of DAWG

Proof.

- Because the DAWG has at most $2n$-1 nodes, the spanning tree T contains at most $2n$-2 edges.

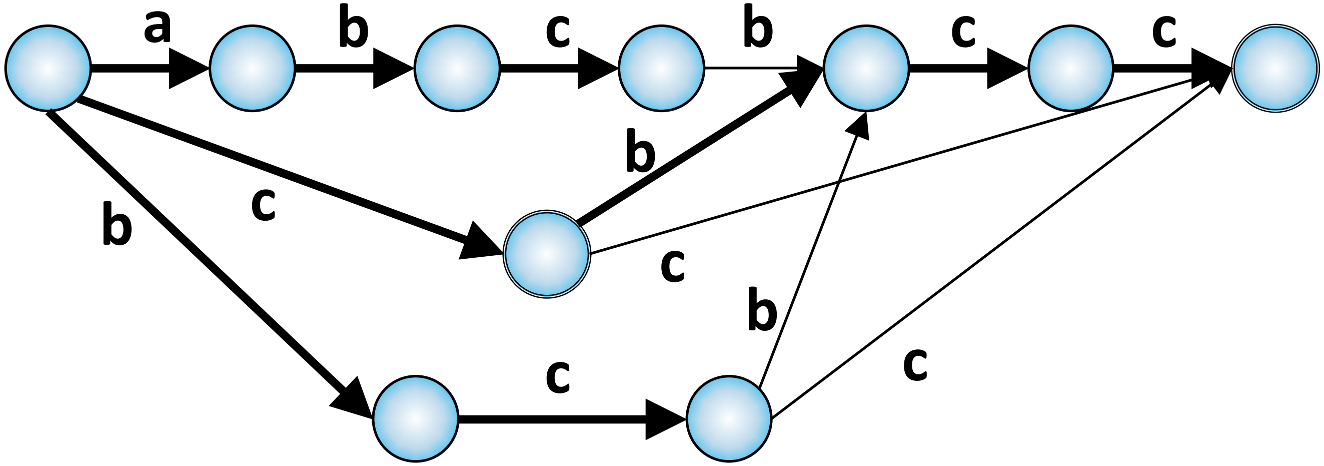
  Next, we count the number of edges outside T.



Number of Edges of DAWG

Proof.

- For any edge e outside T, we consider path xay, where x is the path from the root to e, a is the label of e, and y is any path after e such that xay is a suffix of w.



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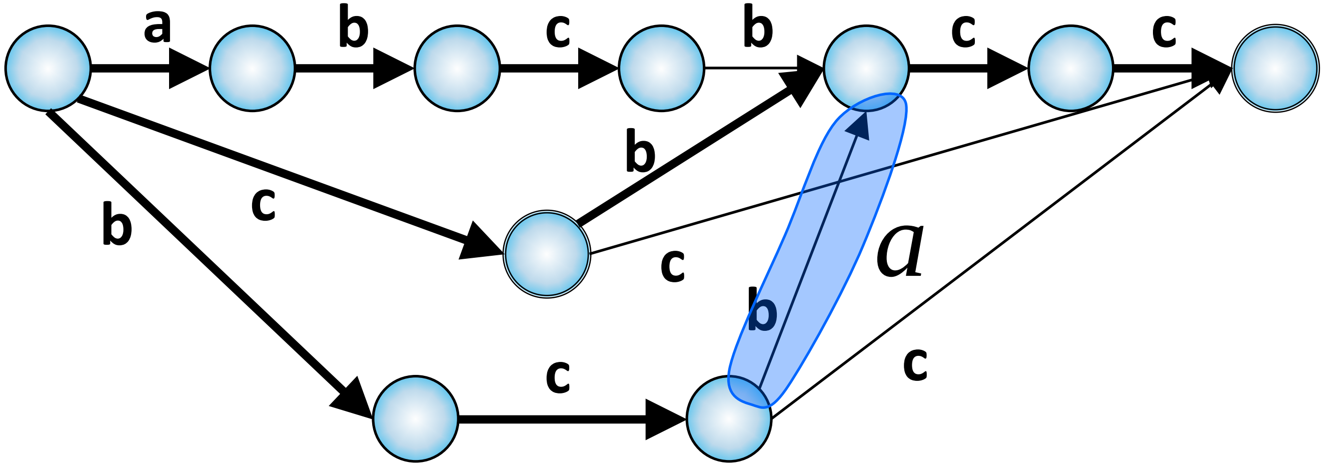

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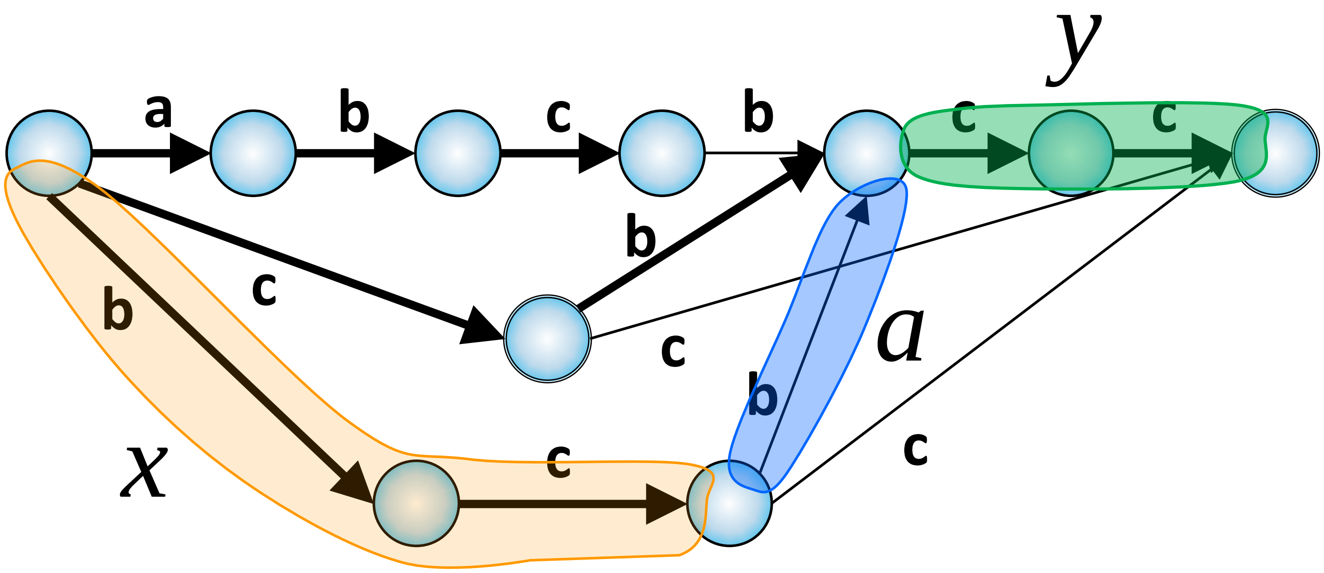
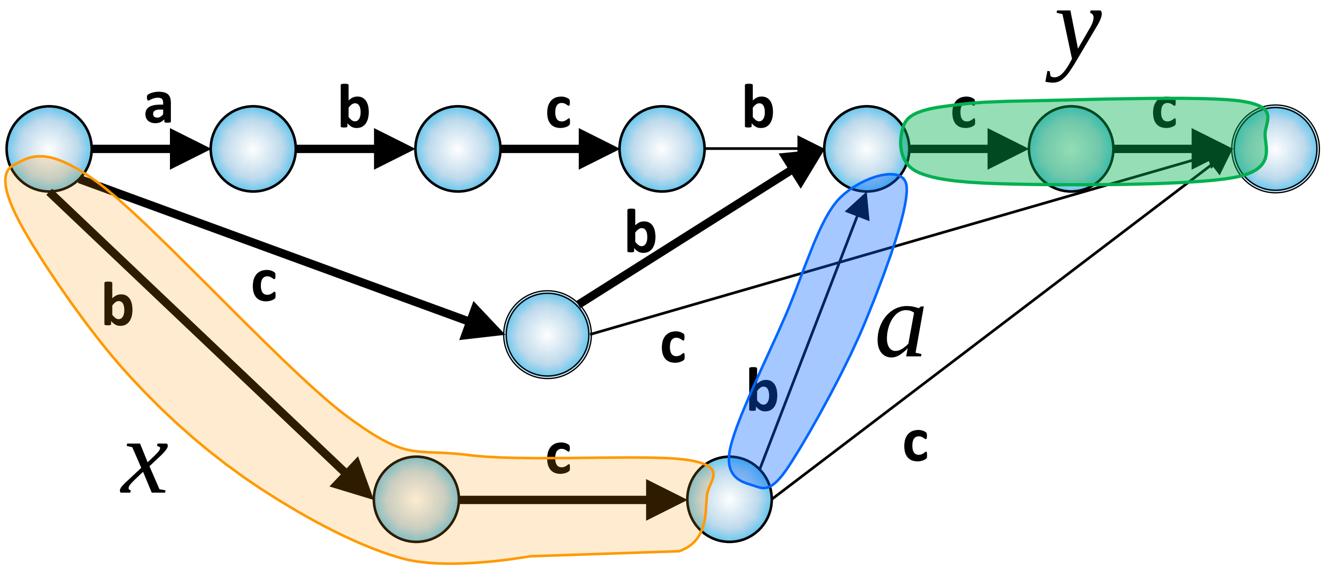


Number of Edges of DAWG

Proof.

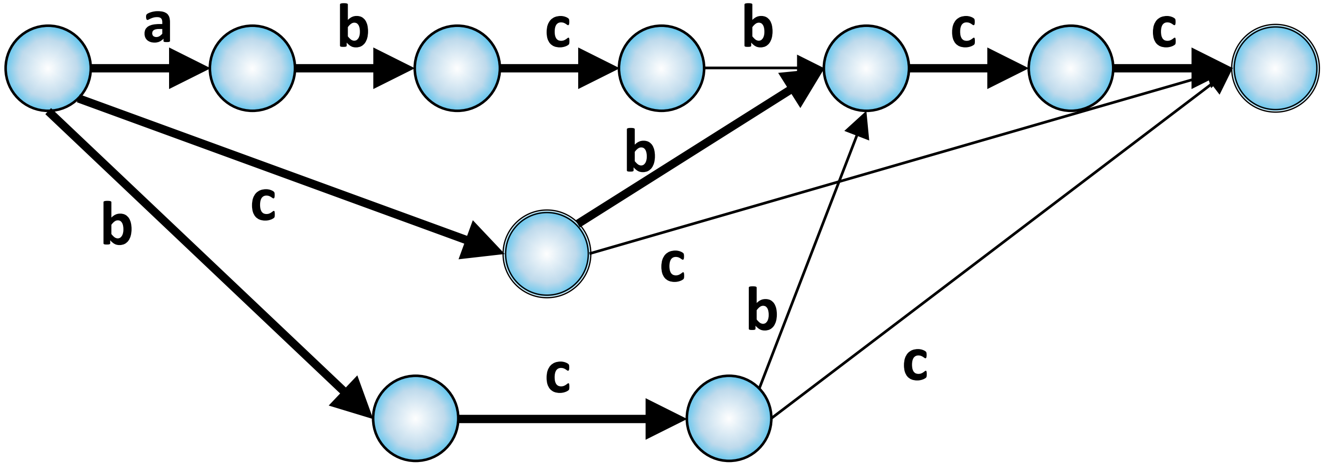
- This maps the edge e to the suffix xay of w. Moreover, any other edge outside the spanning tree T cannot be mapped to the same suffix xay.



Number of Edges of DAWG

Proof.

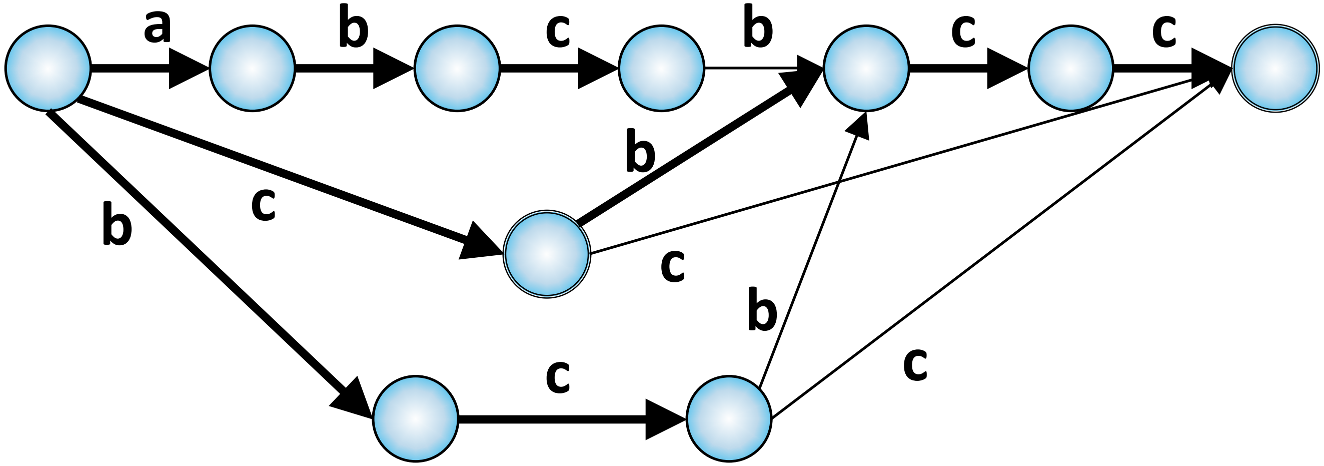
- Hence the mapping is injective.
  Since the spanning tree contains at least one suffix of w, there can be at most n-1 suffixes to which the edges outside T can be mapped.



Number of Edges of DAWG

Proof.

- Therefore, the number of edges outside T is at most n-1. Overall, DAWG has at most $(2n-2)+(n-1) = 3n-3$ edges.



The size of DAWG

Theorem 3

The DAWG of any string of length n has at most $2n$-1 nodes and $3n$-3 edges.

- Note: The bound for the number of edges can further be shaved to $3n$-4, and it is tight, i.e., the DAWG of string $ab^{n-2}c$ contains $3n$-4 edges.

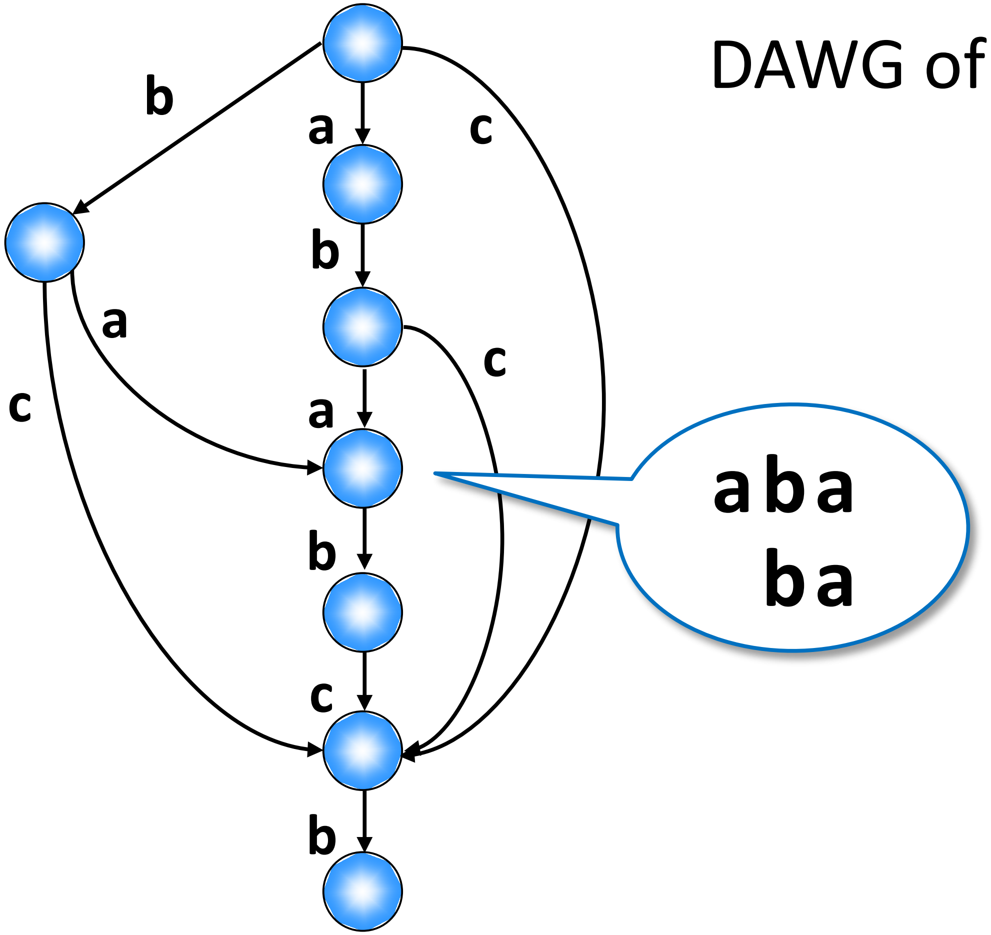
Properties of DAWG nodes

Lemma 3

Two strings x and y are represented by
the same node of the DAWG of w
iff x and y end at the same positions in w.

- This is true because we merged nodes of the suffix trie of w iff they have isomorphic subtrees (hence, the same sets of ending positions).

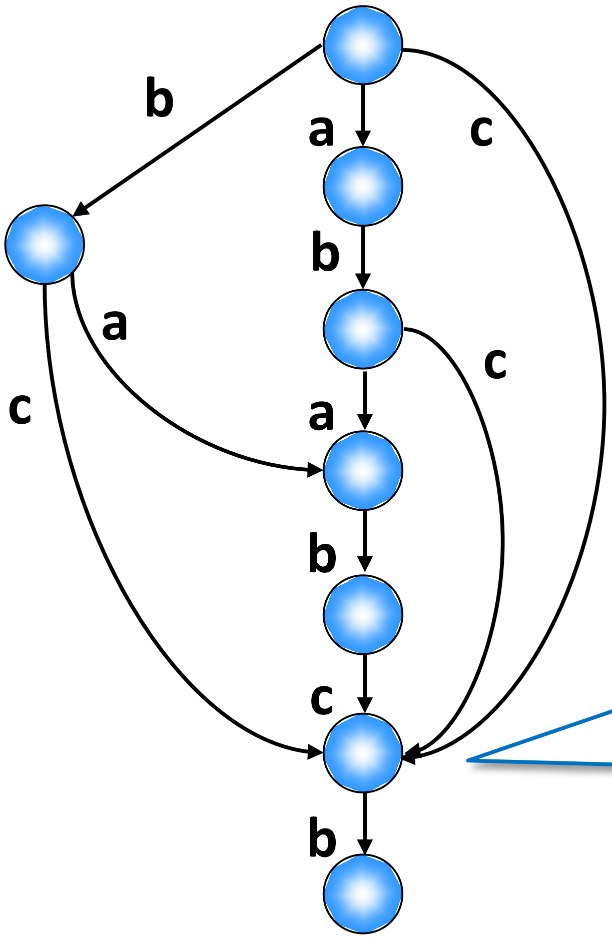
Properties of DAWG nodes



aba
ba

1 2 3 4 5 6
DAWG of **ababcb**

Properties of DAWG nodes



1 2 3 4 5 6

DAWG of **ababcb**

ababc

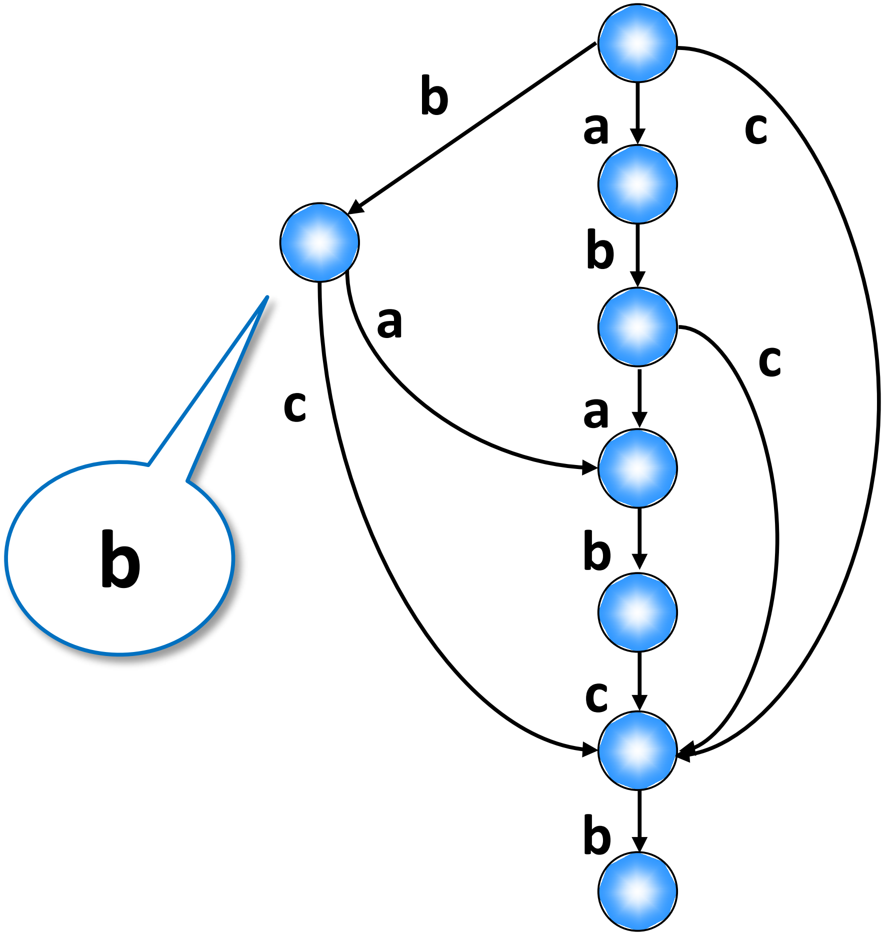
babc

abc

bc

c

Properties of DAWG nodes



1 2 3 4 5 6

DAWG of **ababcb**

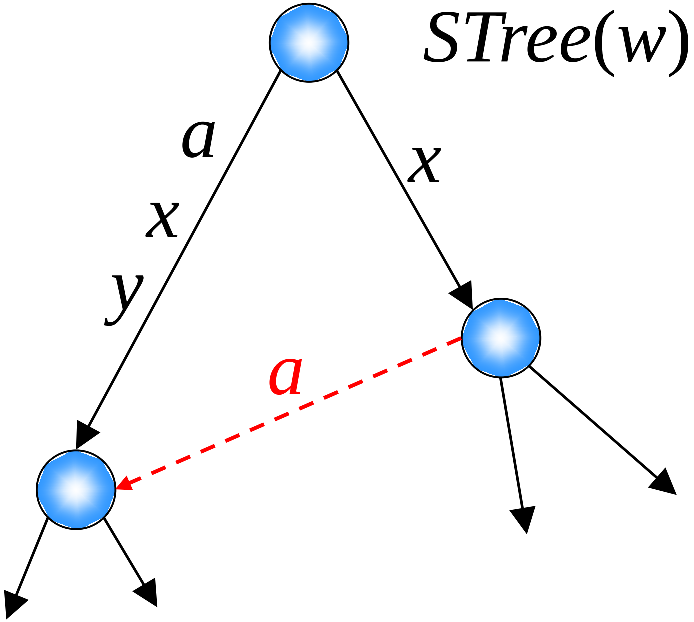
Connection with Weiner Links

Corollary 2

The DAG consisting of the explicit and implicit Weiner links of the suffix tree of string w is the DAWG of the reversed string $\boldsymbol{w^R}$.

Connection with Weiner Links

w		x		a	x	y		a	x	y	



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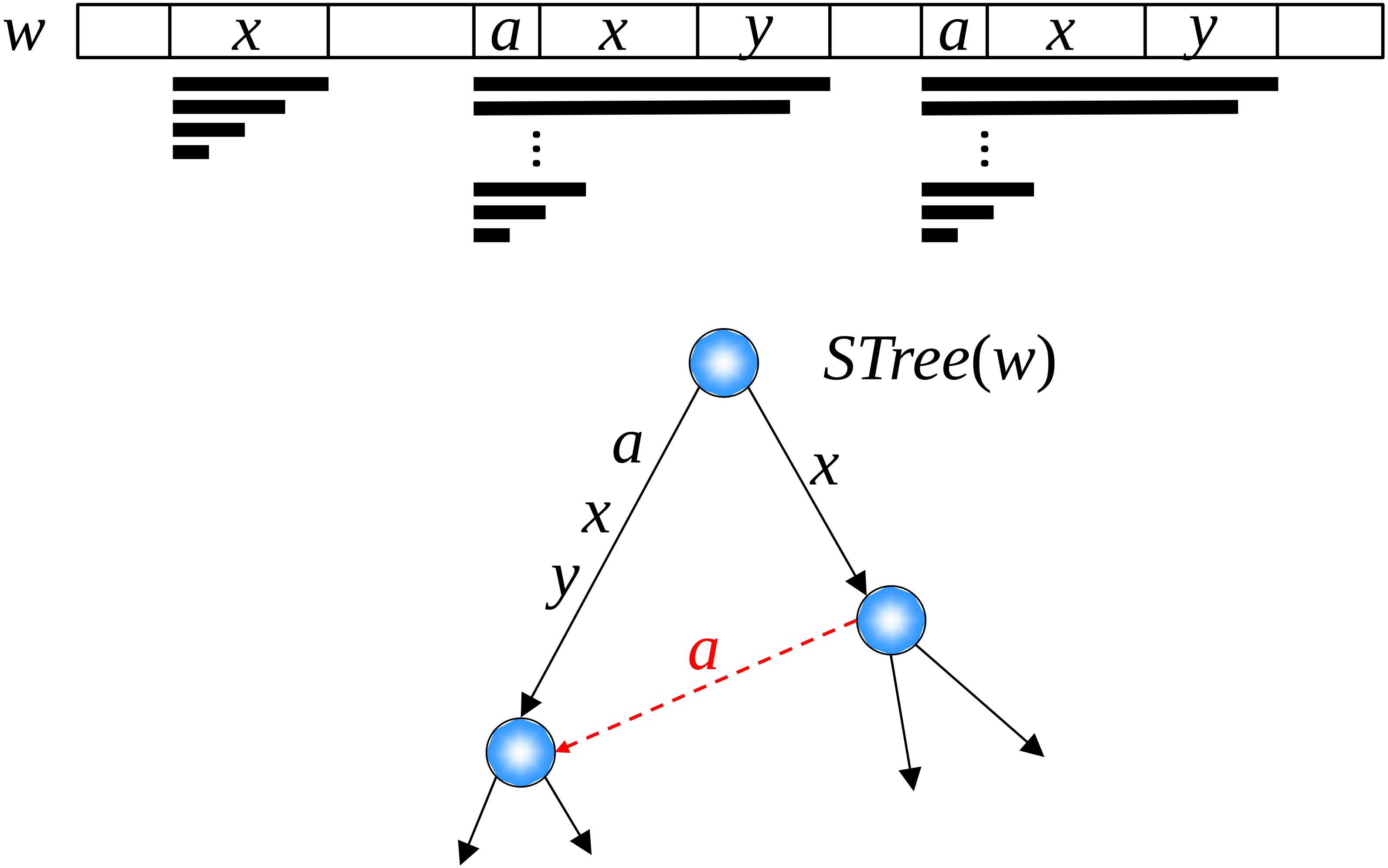
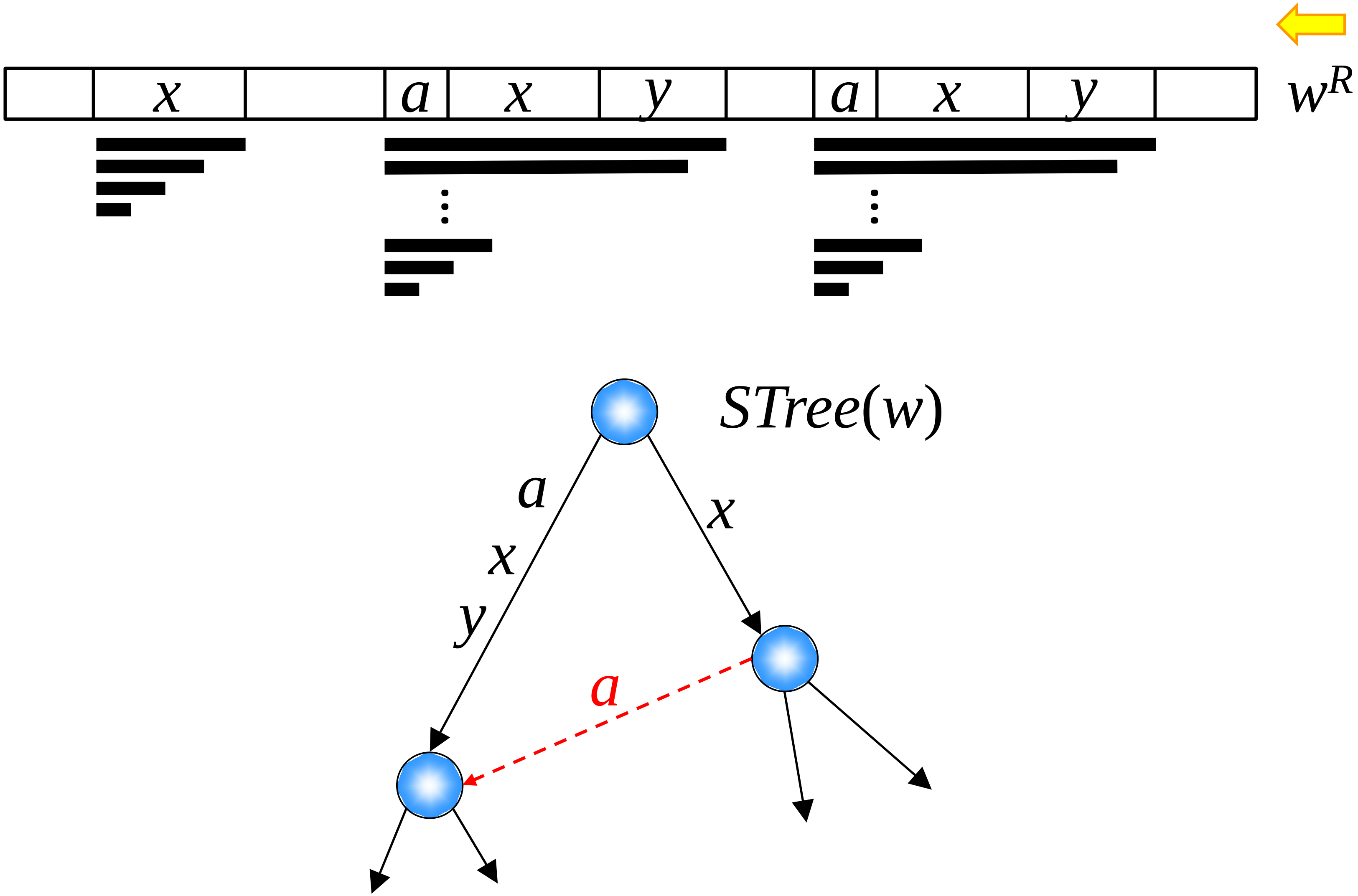


Connection with Weiner Links



Applications of DAWGs (Incomprehensive)

- Bidirectional pattern matching [Folklore]
- Approximate pattern matching
  [Ukkonen & Wood, 1993]
- Pattern matching with variable-length don't cares
  [Kucherov & Rusinowitch, 1997]
- Finding minimal absent words
  [Crochemore et al., 1998, 2015, Fujishige et al. 2016]
- Compact online Lempel Ziv 77 factorization
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- Finding α-gapped repeats [Tanimura et al., 2015]
- Finding maximal-exponent substring in
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Minimal Absent Words

- A string y is said to be a **minimal absent word** (**MAW**) of a string w, if
  1. y does not occur in w, but
  2. proper substrings of y occur in w.

- E.g.) If w = **abaab** and Σ = {**a**, **b**}, then the MAWs of w are **aaa**, **aaba**, **bab**, **bb**.

- MAWs can be used to build phylogeny [Chairungsee & Crochemore, 2012].

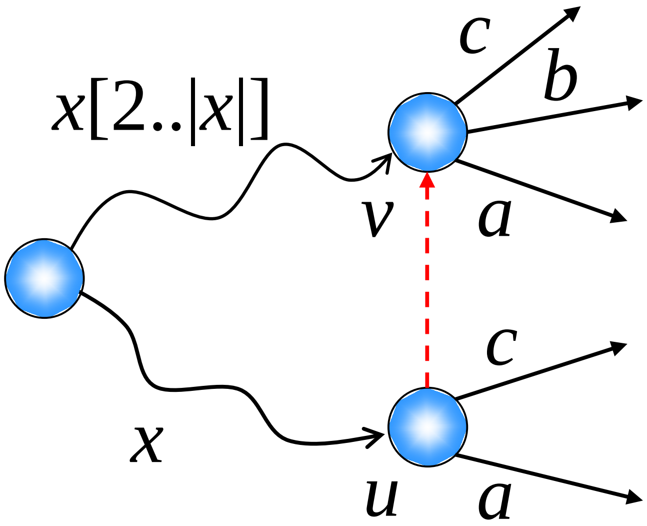
Minimal Absent Words

- For a node u of the DAWG of w, let x be the shortest string represented by u.

- Let $v = \mathrm{suf\_link}(u)$.

- Then, xb ($b \in \Sigma$) is a MAW of w iff
  1. there is no out-edge from u labeled b, and
  2. there is an out-edge from v labeled b.

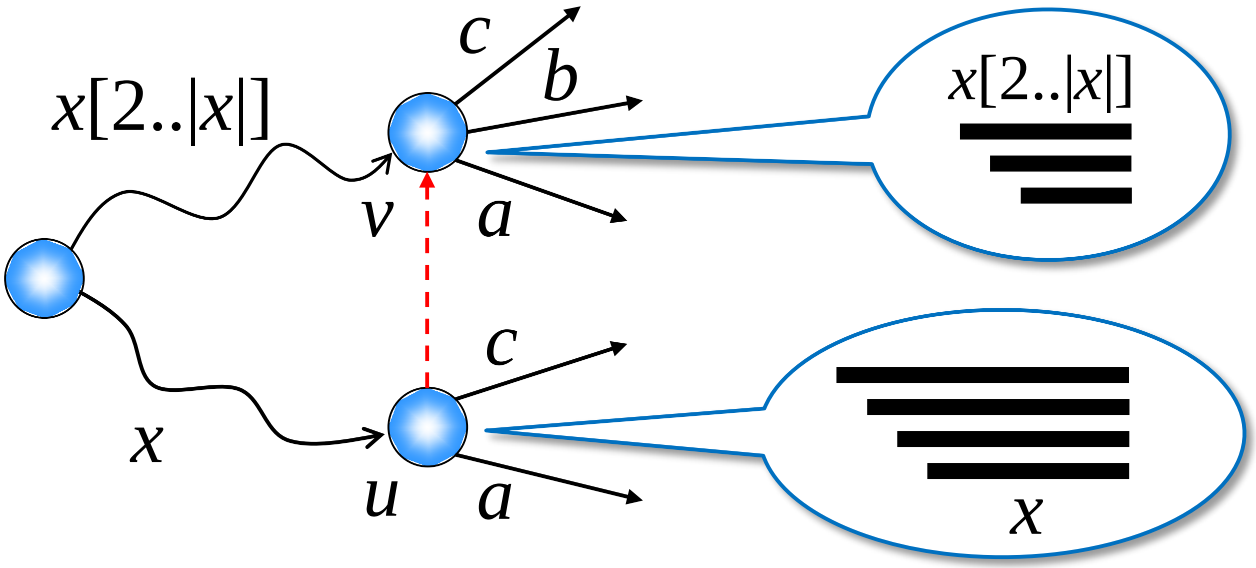
Minimal Absent Words



- xb does not occur in w, and
- both x and $x[2..|x|]b$ occur in w.

$\Leftrightarrow$ xb is a MAW of w.

Minimal Absent Words



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MAW Computation with DAWG

Theorem 4

Using the DAWG of string w, we can compute all MAWs of w in $O(\sigma n)$ time.

$\sigma = |\Sigma|, \ n = |w|$

- For each node of the DAWG of w, it is sufficient to test at most σ letters.
- The DAWG of w has $O(n)$ nodes.

Faster MAW Computation with DAWG

Theorem 5

Using the edge-sorted DAWG of string w, we can compute all MAWs of w in optimal $O(n + |MAW_w|)$ time.

- Testing out-edges of u and v can be charged to either existing edges or MAWs to output.

Direct Construction of DAWGs

- Since the suffix trie of string of length n can contain $\Omega(n^2)$ nodes, converting the suffix trie into the DAWG takes $O(n^2)$ time.
- Can we construct DAWGs directly?

Online Construction of DAWGs

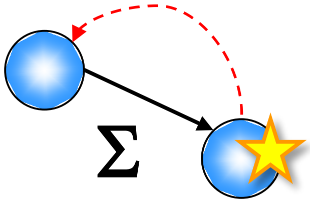
Theorem 6

The DAWG of a given string w of length n can be constructed online in $O(n \log \sigma)$ time, where σ is the alphabet size.

- We incrementally build the DAWG of $w[1..i]$ for increasing $i = 1, \ldots, n$ (left-to-right online).
- The DAWG is annotated with suffix links.
- The $\log \sigma$ factor is the cost to sort and search branching edges.

Online Construction of Suffix Tries

- Before going to online construction of DAWGs, we consider online construction of suffix tries.



ababcb

When a new letter $w[i]$ arrives, we begin with the node which represents $w[1..i\text{-}1]$.

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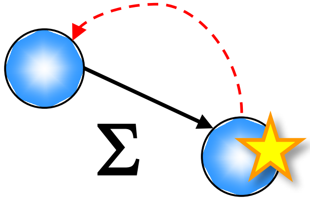

ababcb

If we cannot traverse from the star with new character $w[1] = \mathbf{a}$, create a new edge labeled **a**.

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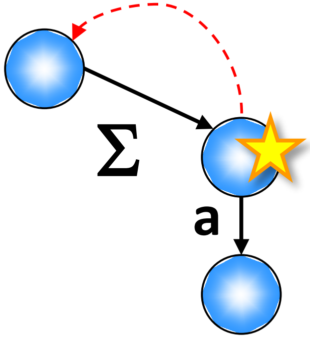

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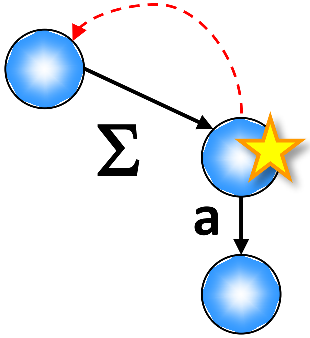

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Then, we move the star via the suffix link, and check whether we can traverse with **a**.

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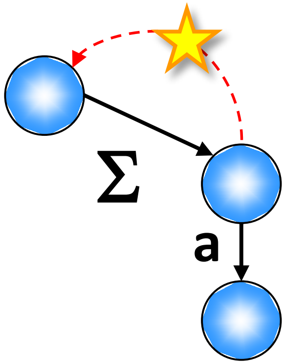

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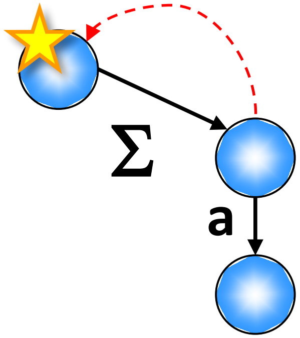

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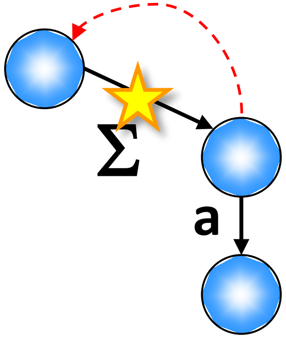

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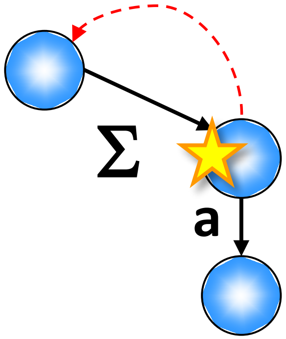

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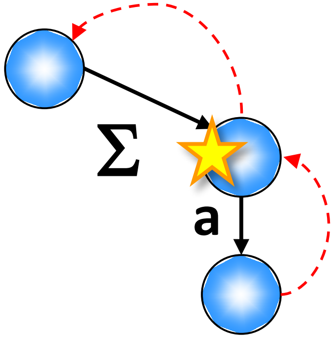

ababcb

After a successful move of the star, we create the suffix link from the new leaf to the node where the star has arrived.

Online Construction of Suffix Tries

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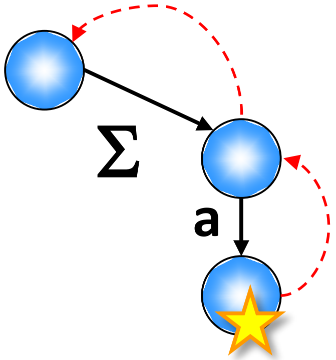

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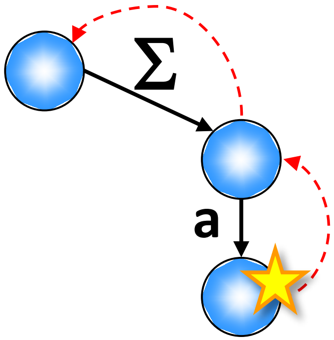
The suffix trie for $w[1..1]$ = **a** is complete.
We move the star to the leaf which represents $w[1..1]$ = **a**.

Online Construction of Suffix Tries



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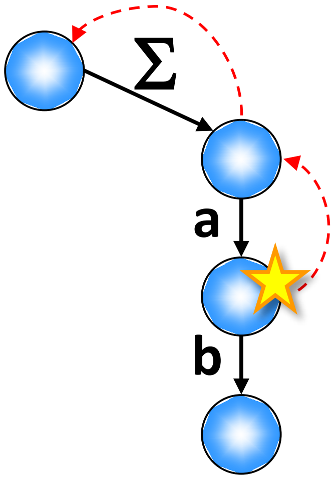

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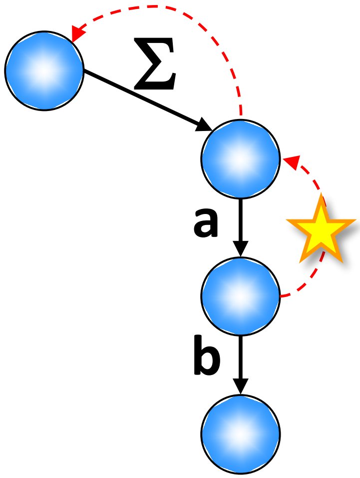

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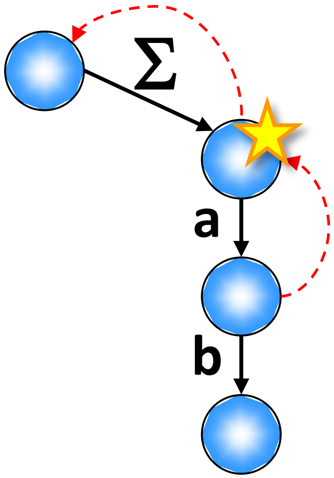

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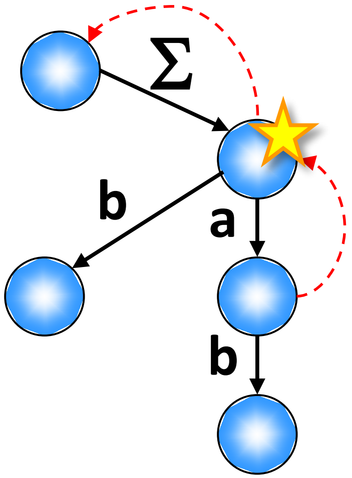

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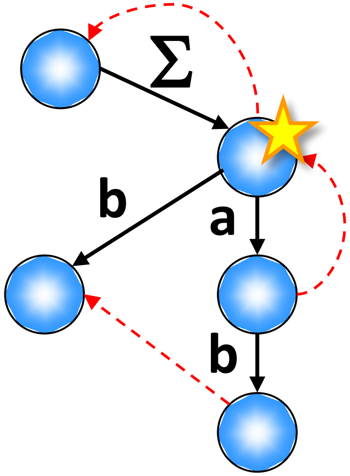
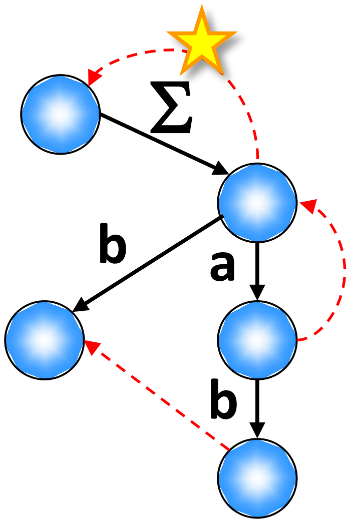


ababcb

We have two new leaves for **ab** and **b**. We create the suffix link from **ab** to **b**.

Online Construction of Suffix Tries



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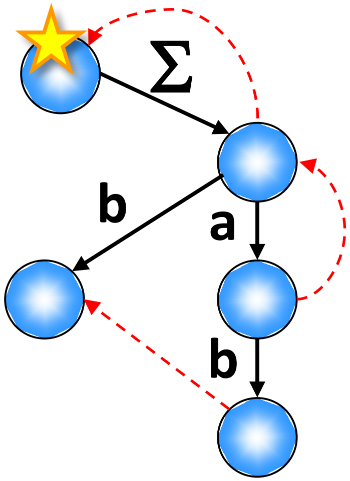

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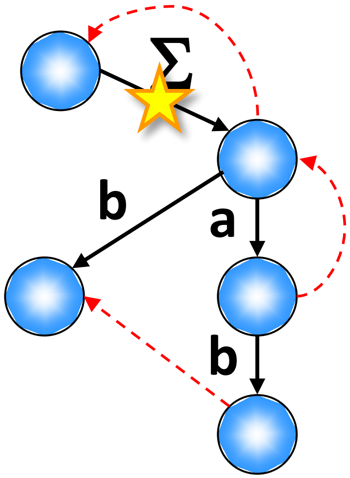

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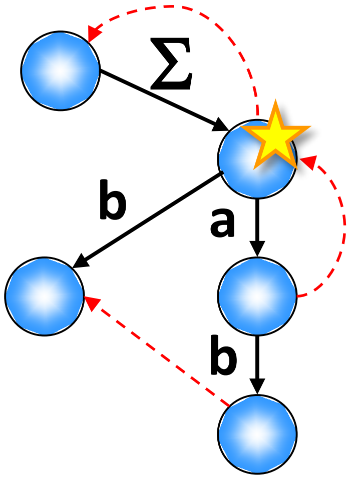

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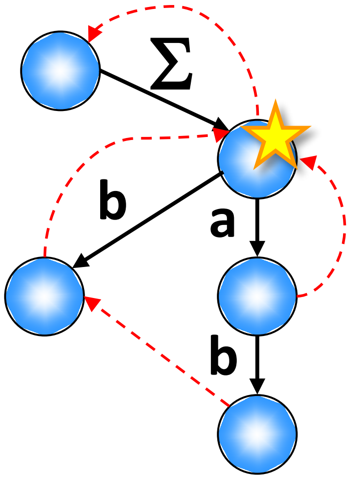

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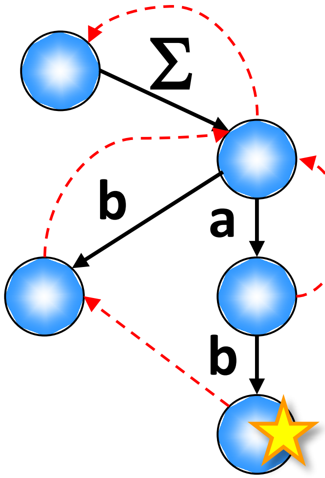

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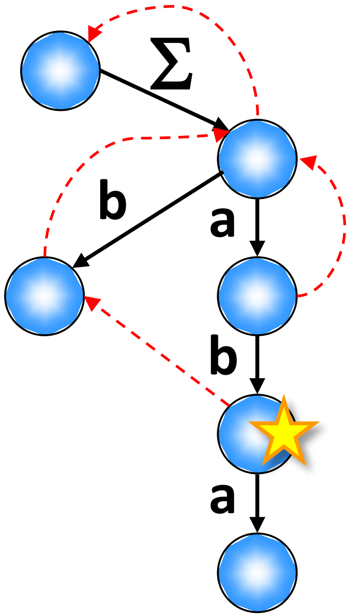

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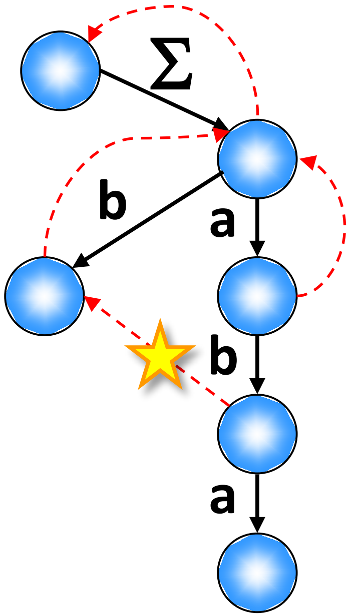

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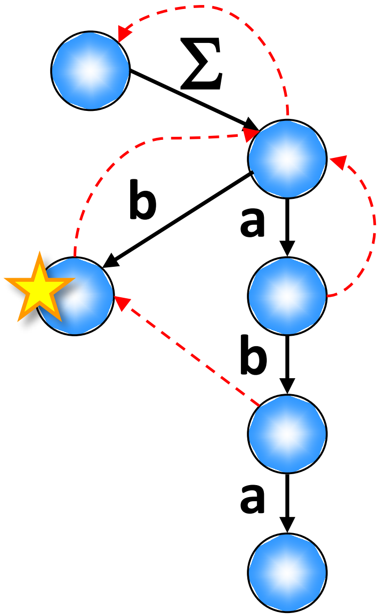

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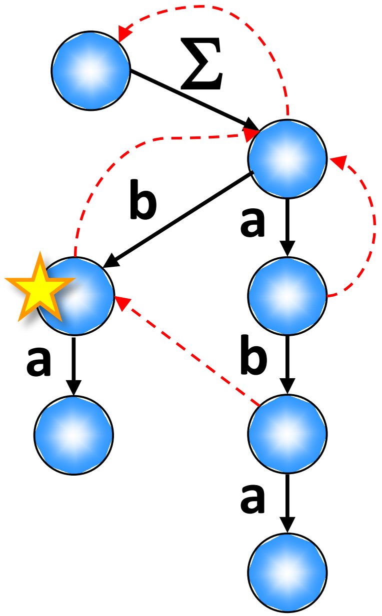

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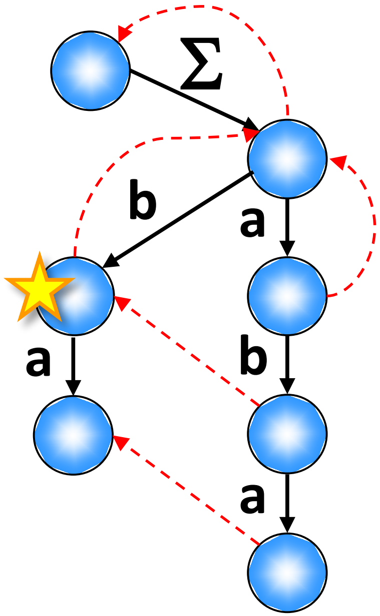

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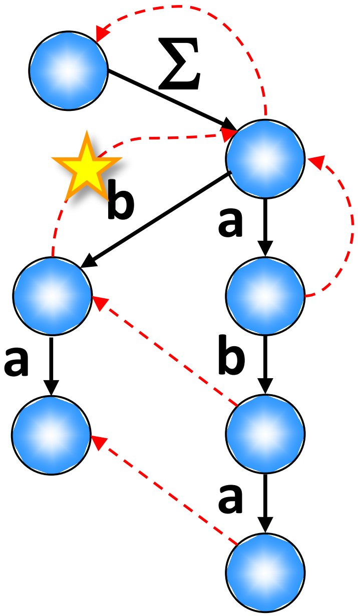

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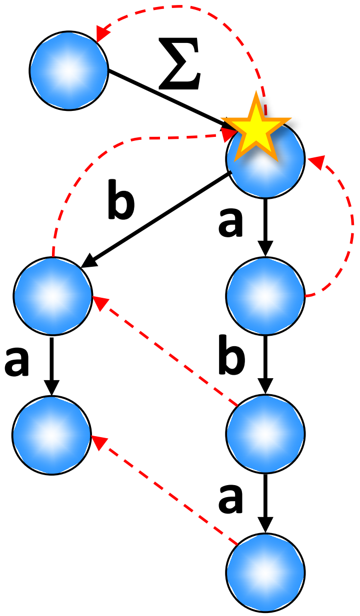

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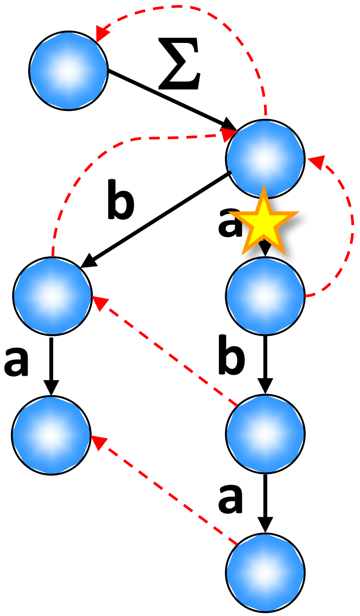
We are looking for an edge labeled $w[3]$ = **a**.
Using a BST for branching edges, we can find it in $O(\log\ \sigma)$ time.

Online Construction of Suffix Tries



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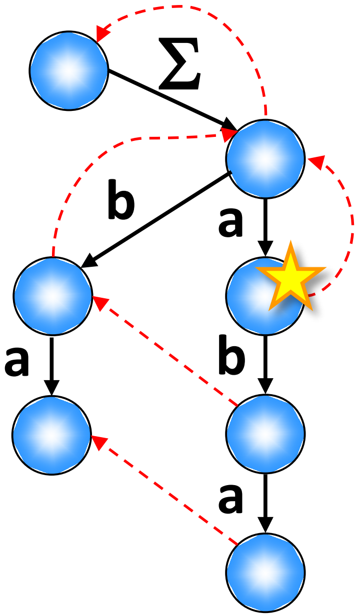

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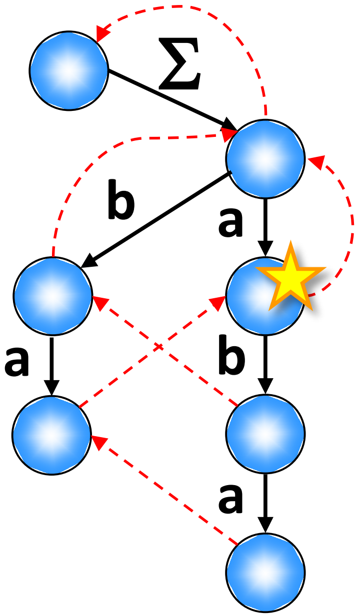

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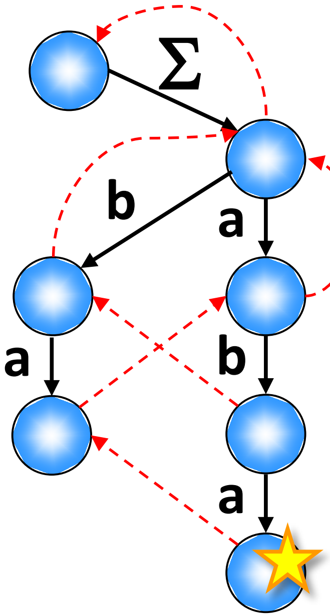

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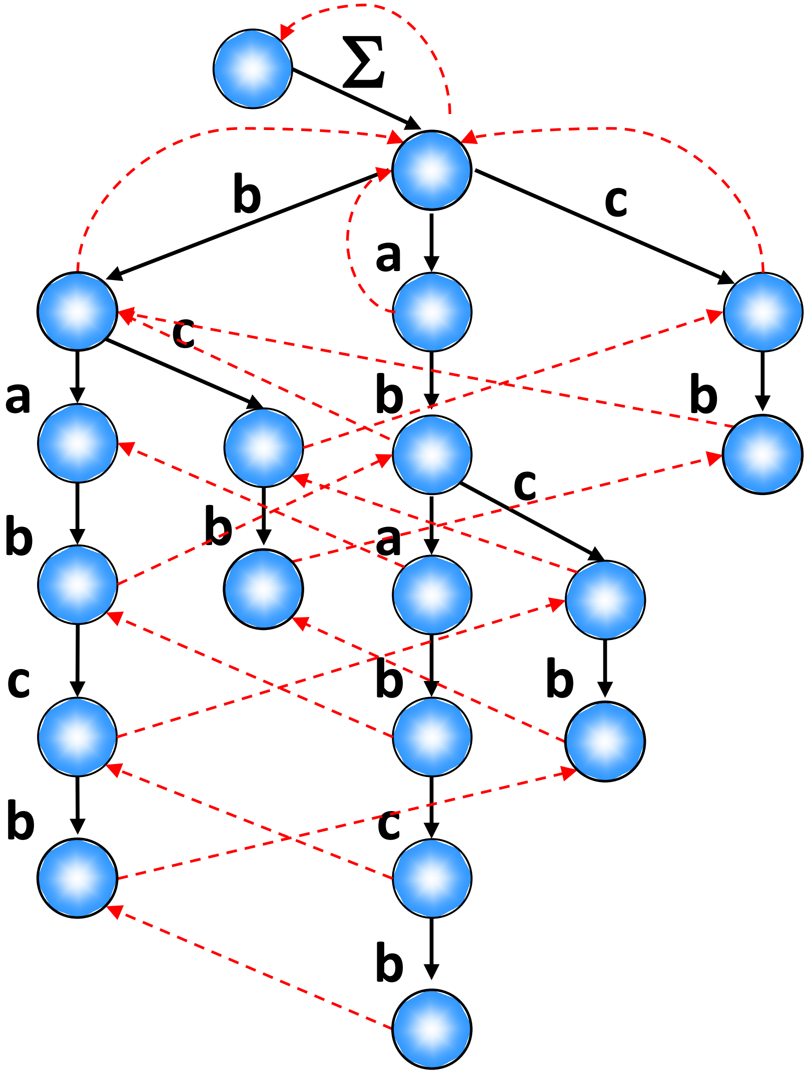

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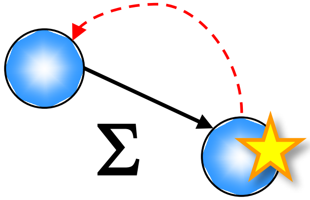
Online Construction of DAWGs

- Online DAWG construction follows the approach of online suffix trie construction.



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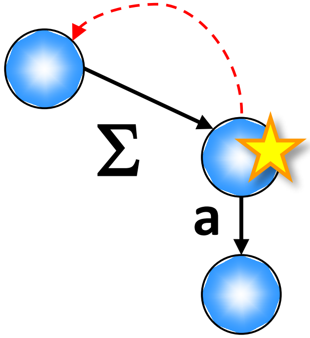

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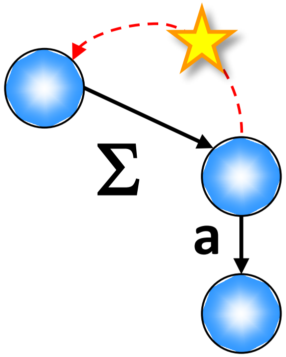

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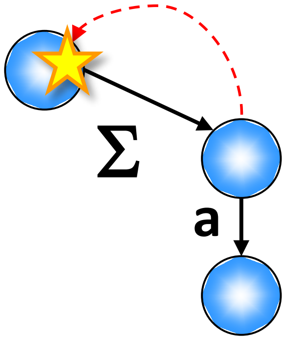

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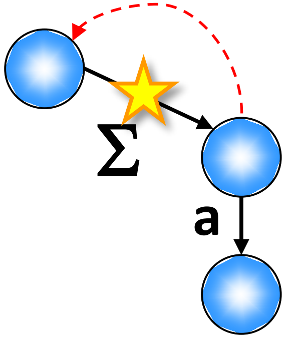

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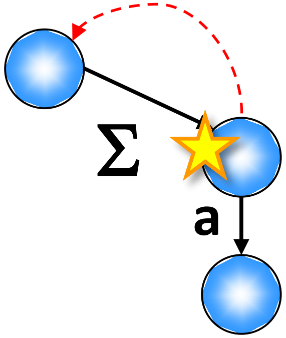

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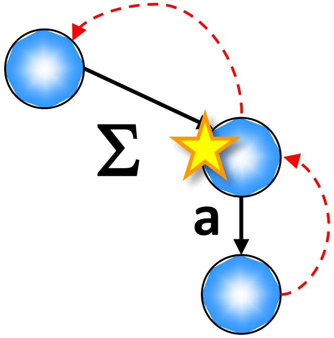

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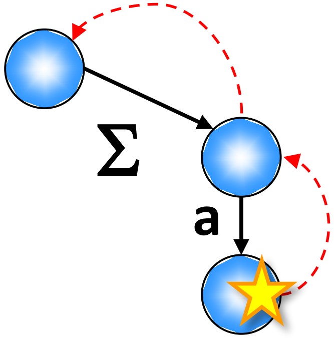

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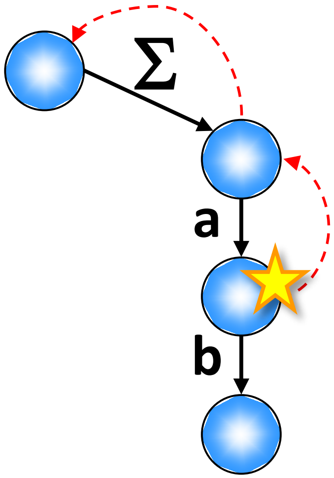

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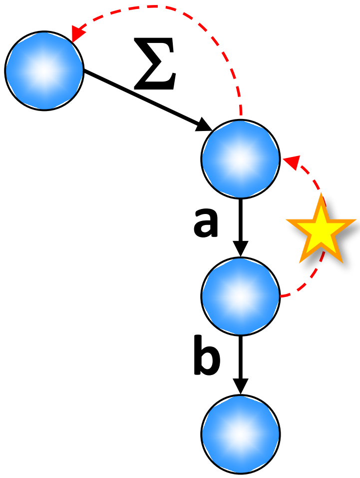

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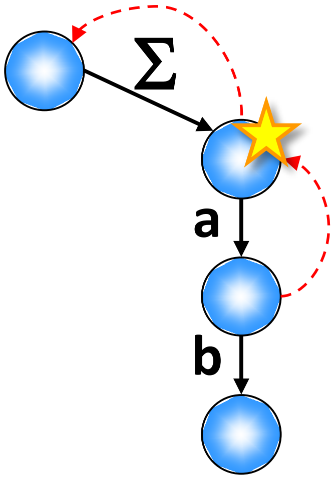

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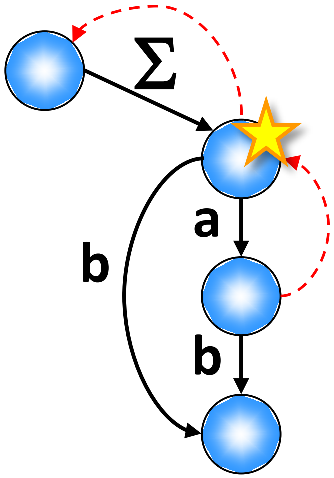

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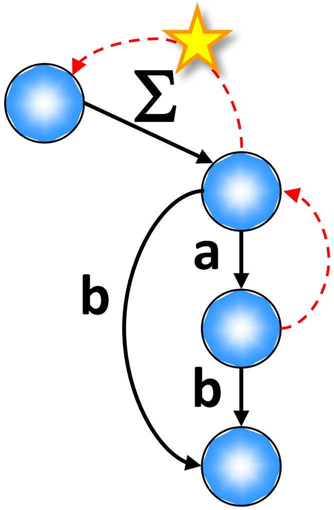
Instead of making a new edge with a new leaf, we create a new edge which leads to the sink node.

This corresponds to merging nodes of the suffix trie!

Online Construction of DAWGs



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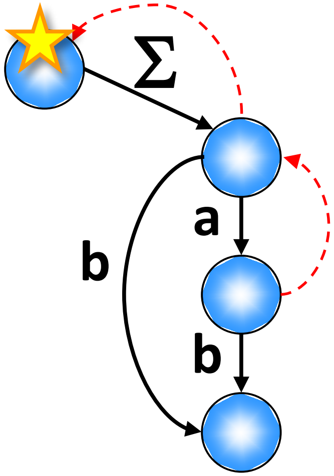

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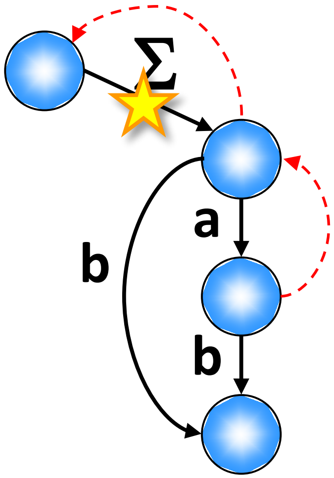

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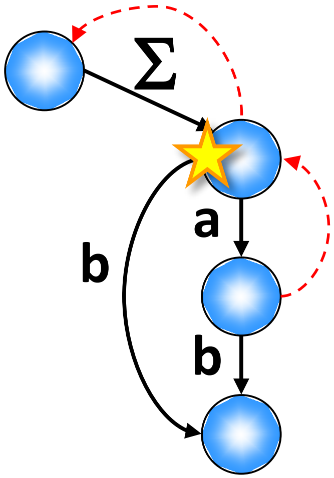

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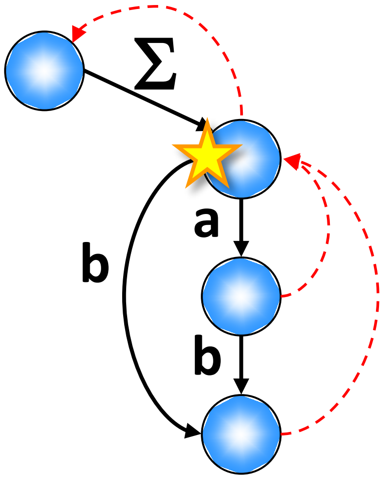

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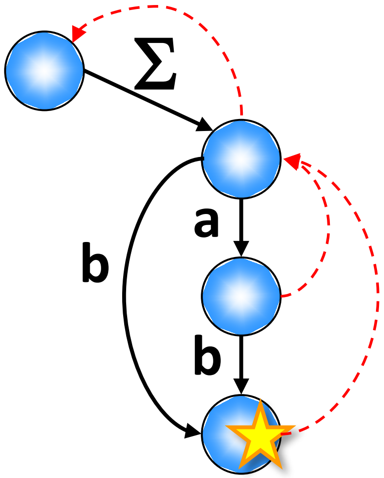
Online Construction of DAWGs



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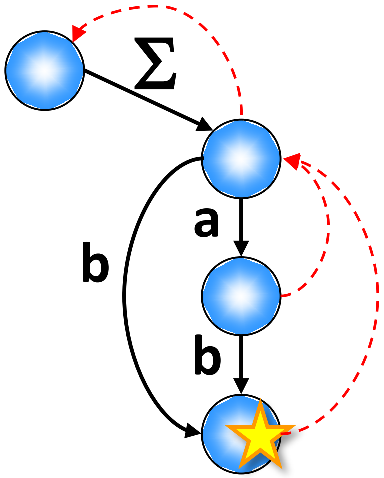

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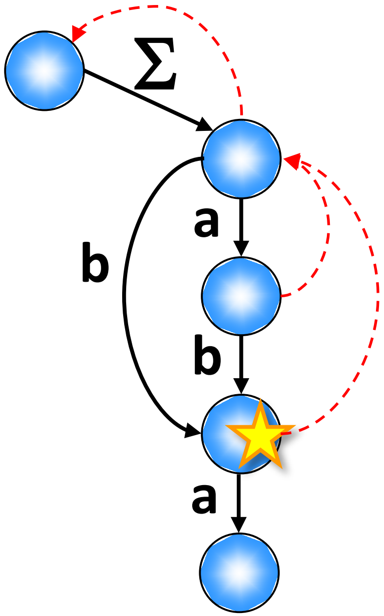

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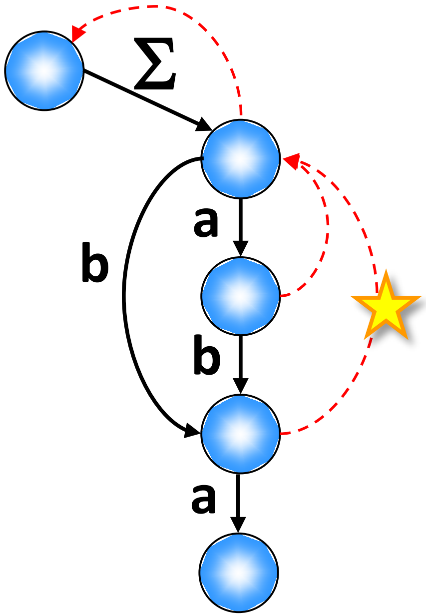

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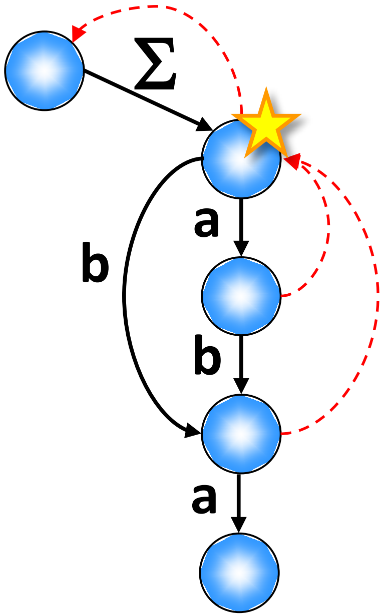

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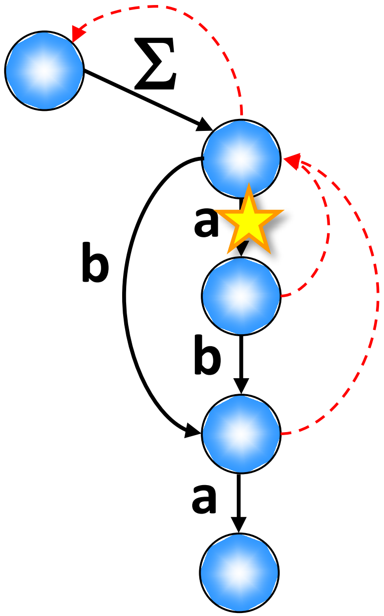

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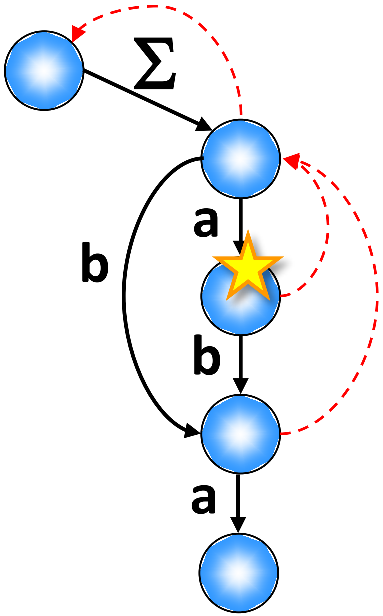

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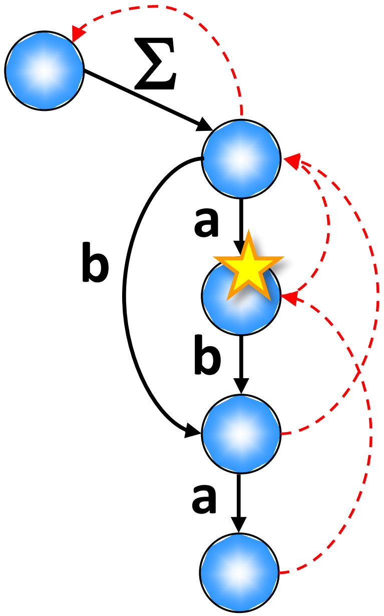

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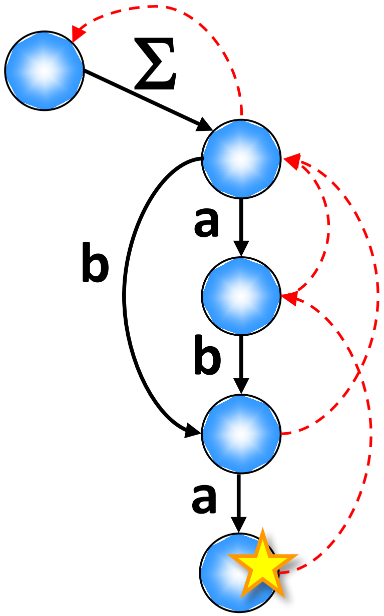

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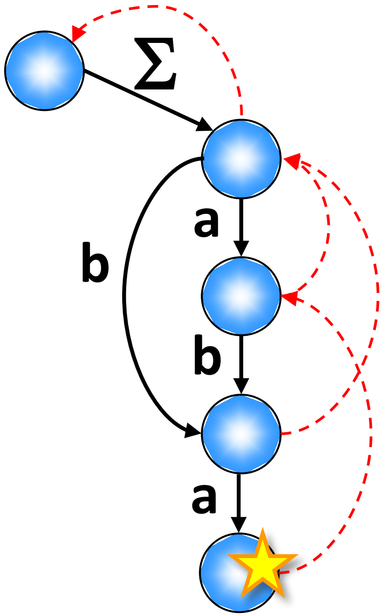

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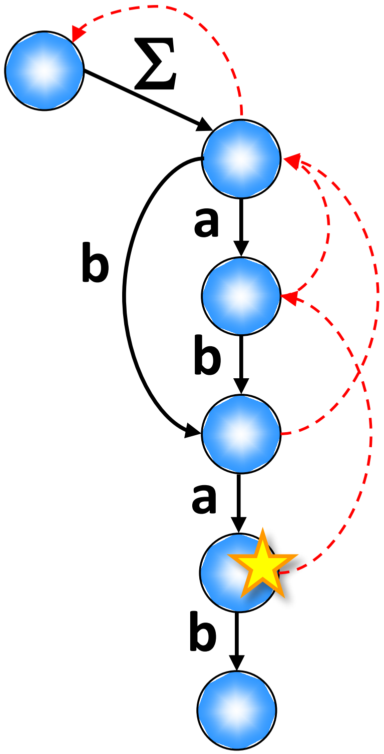

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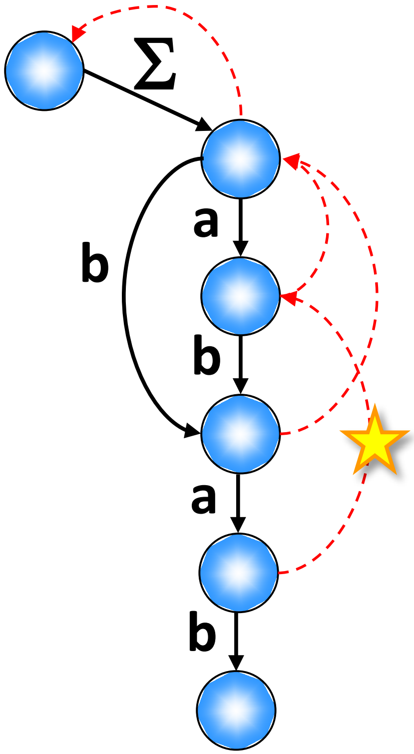

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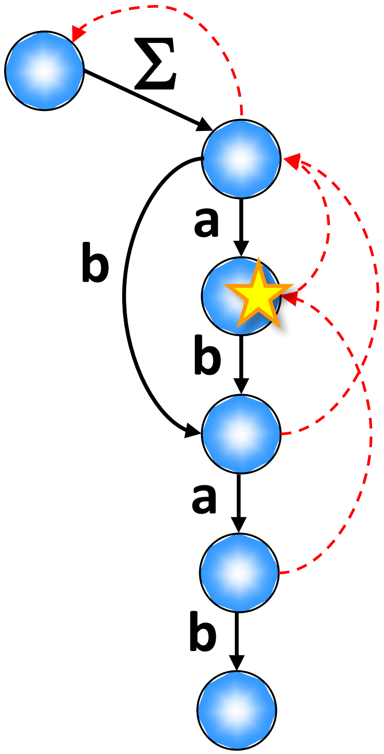

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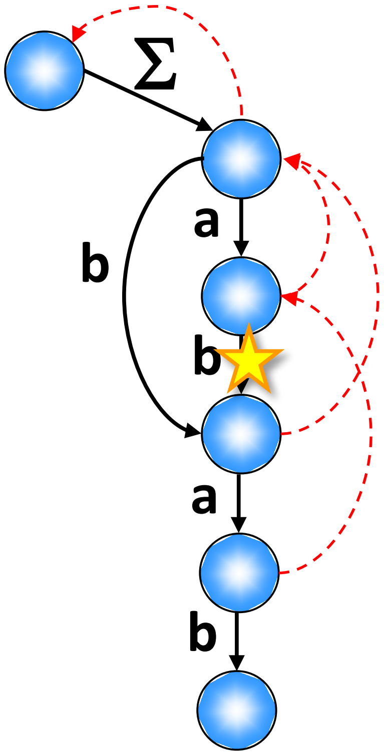

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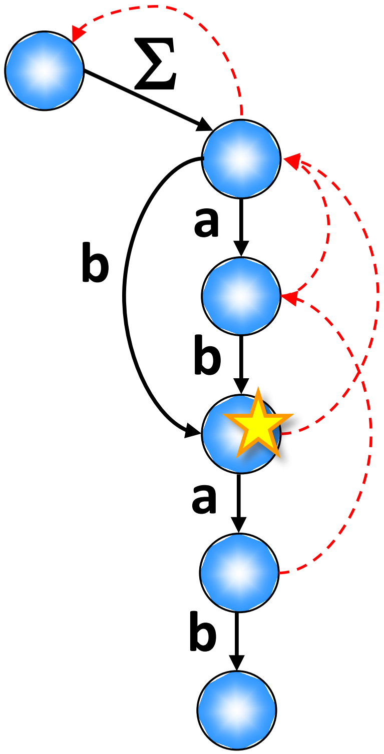

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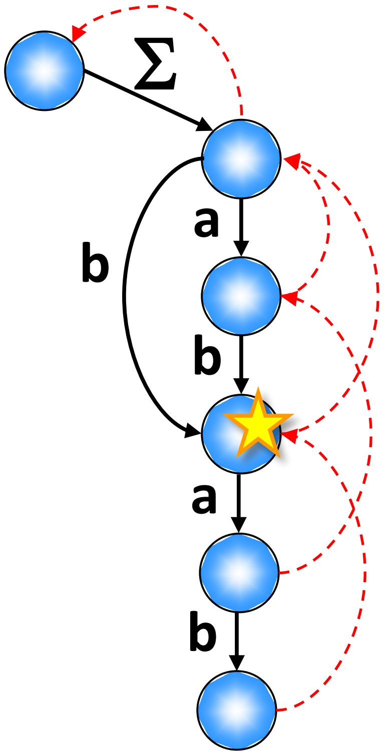

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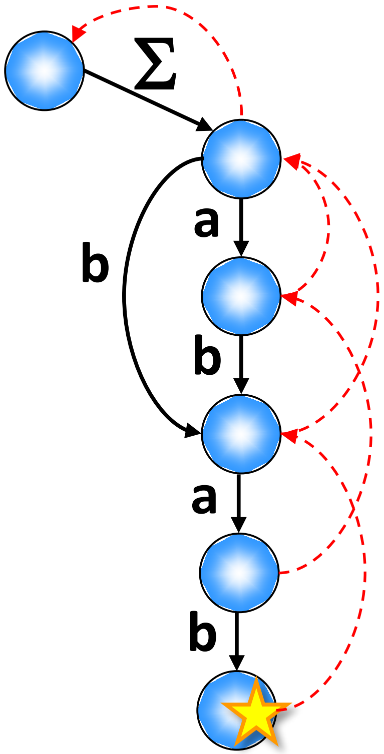

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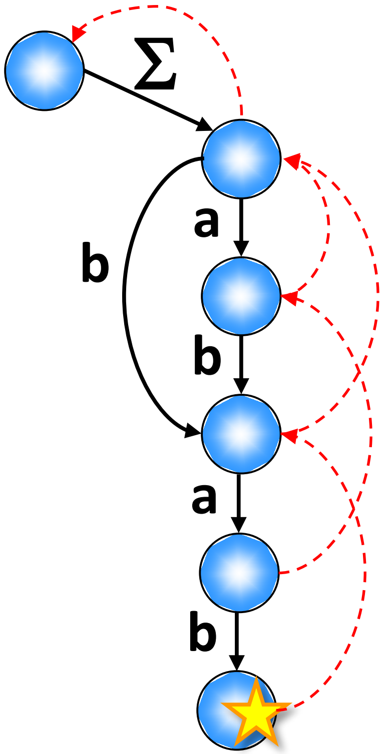

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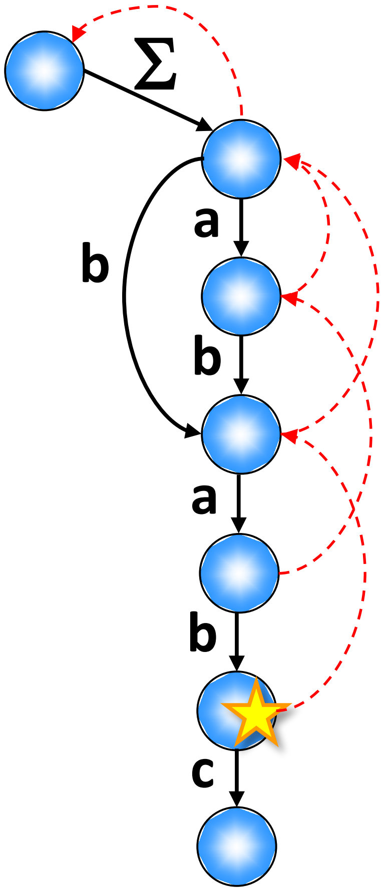

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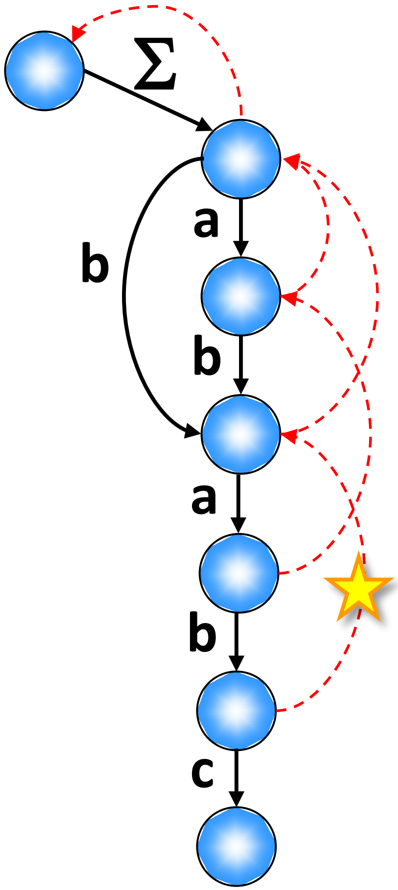

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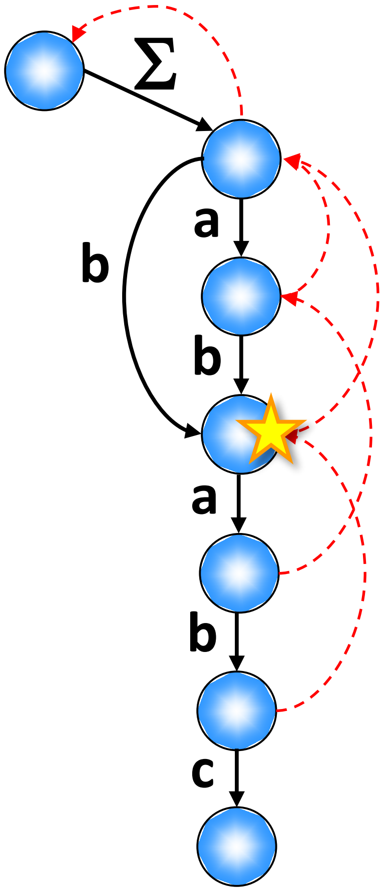

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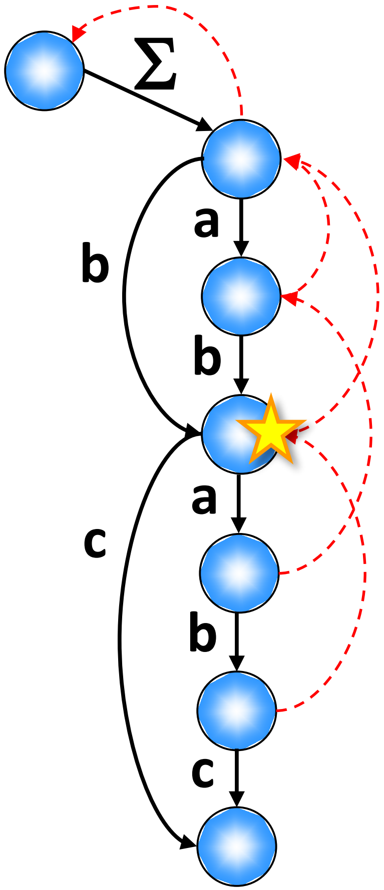

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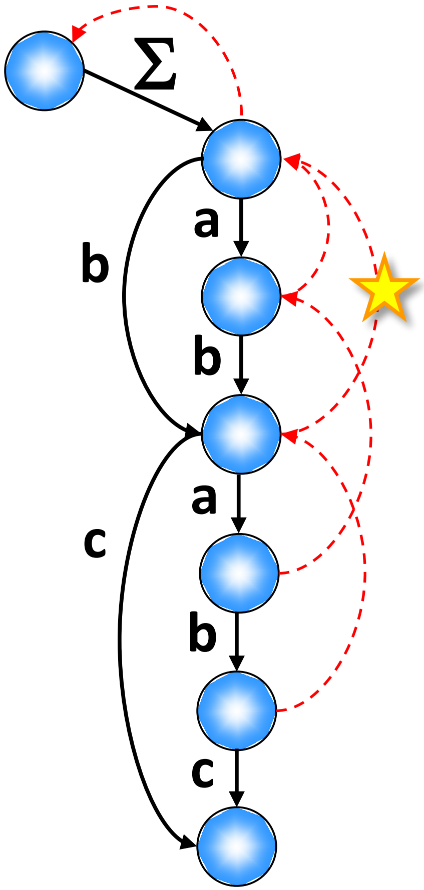

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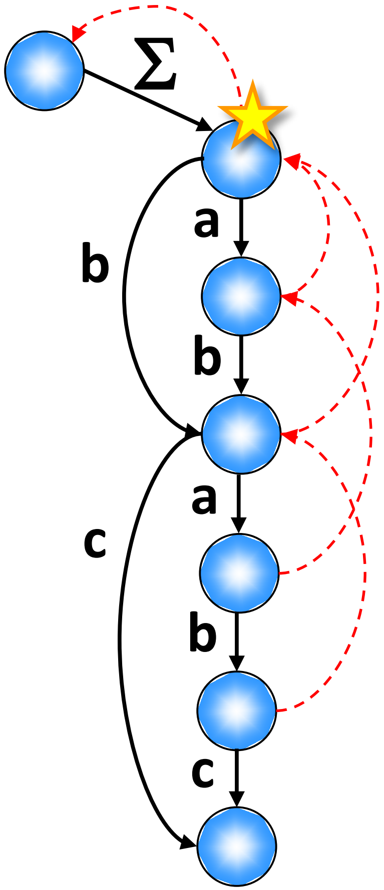

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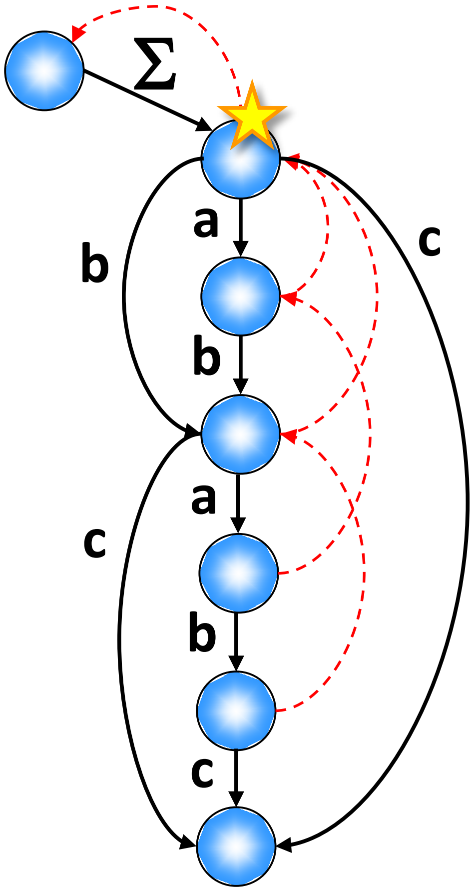

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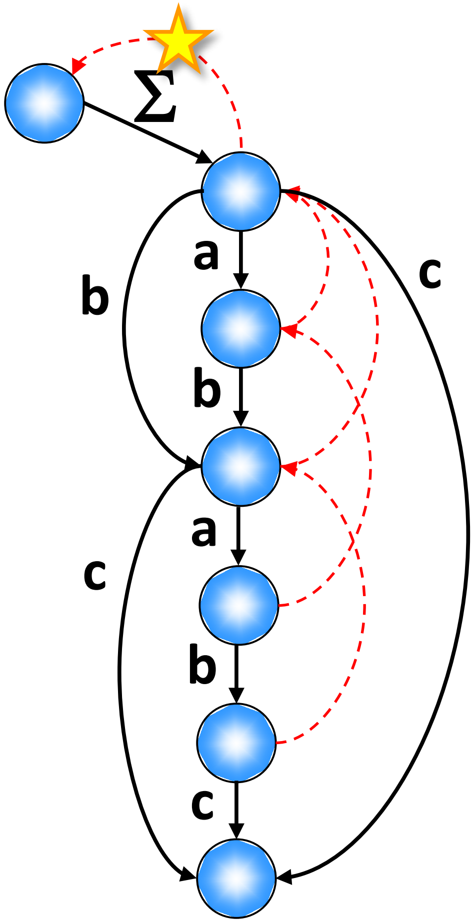

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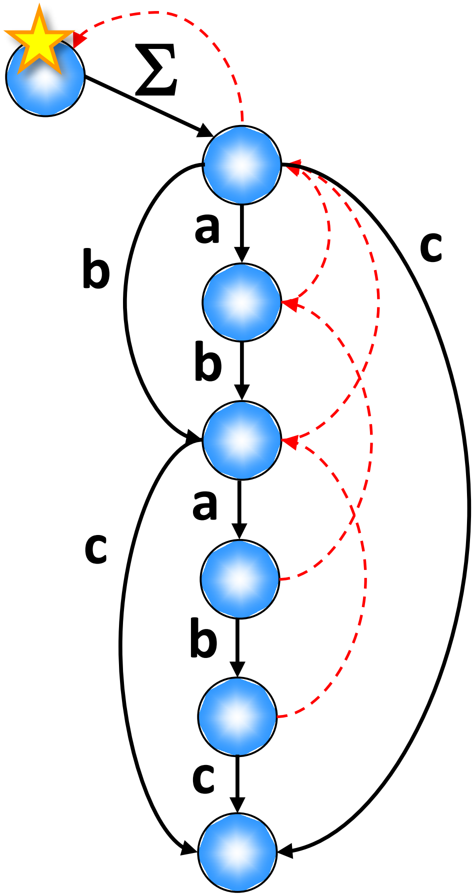

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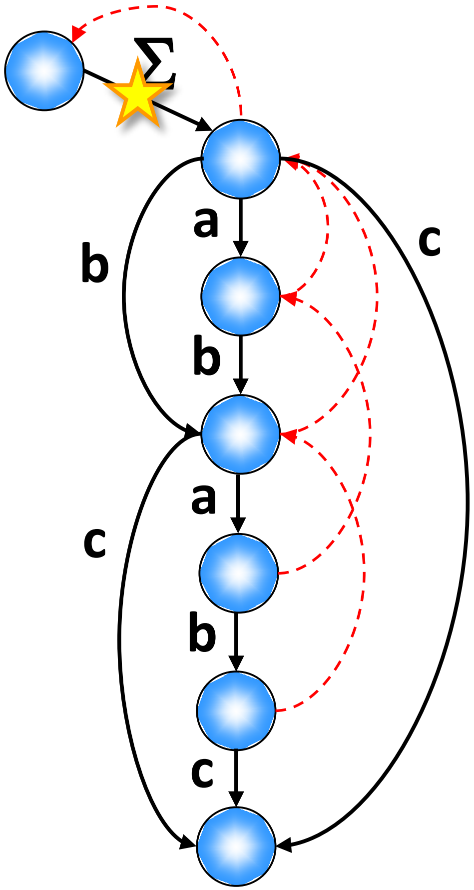

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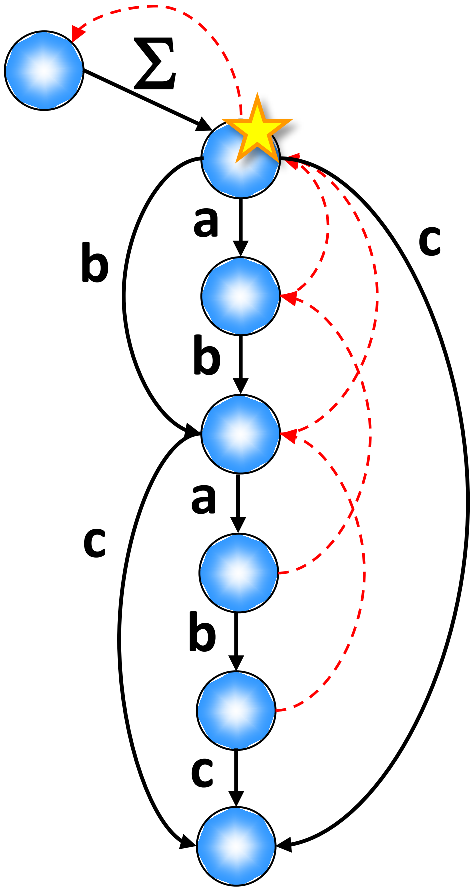

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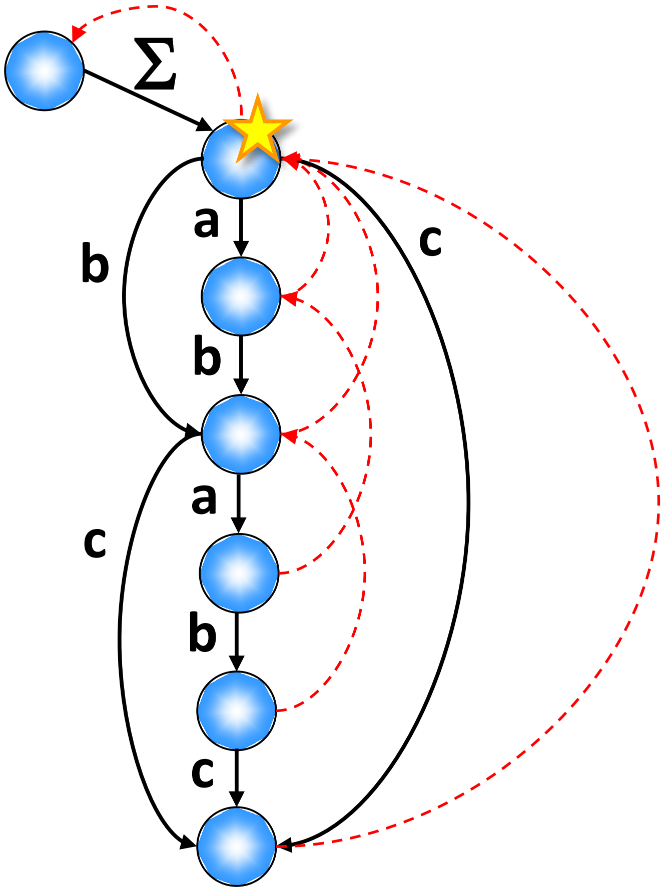

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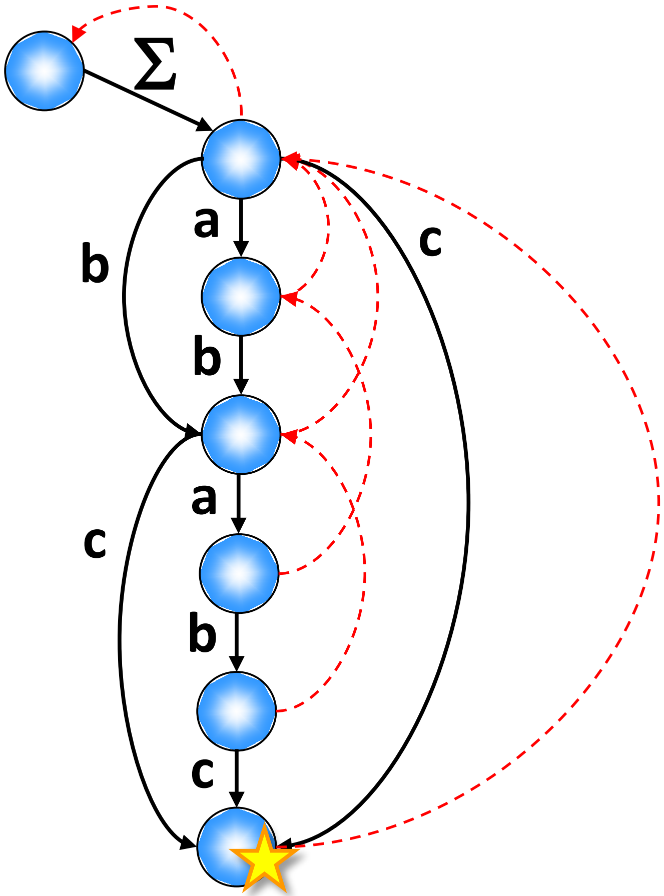

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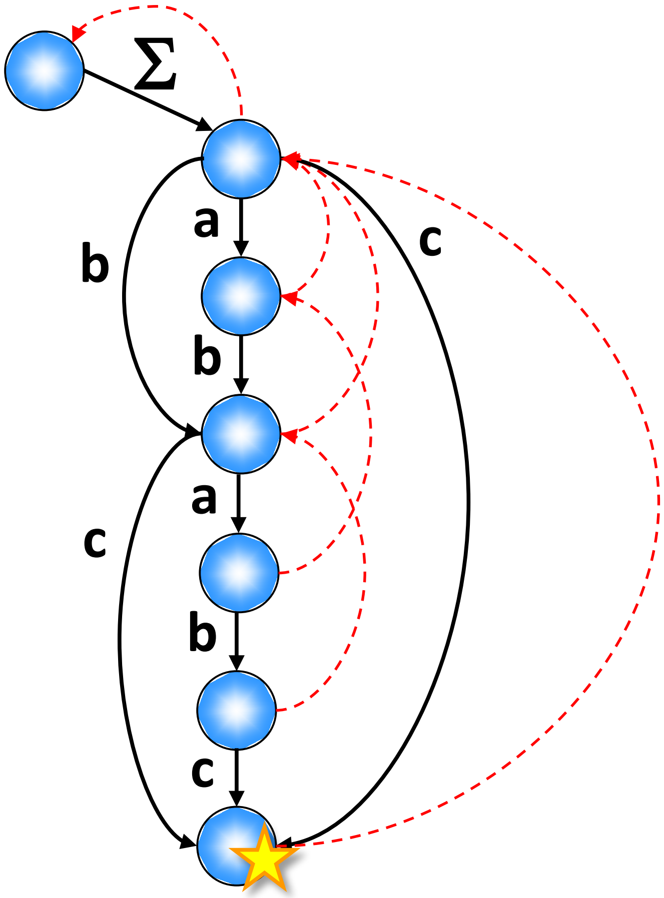

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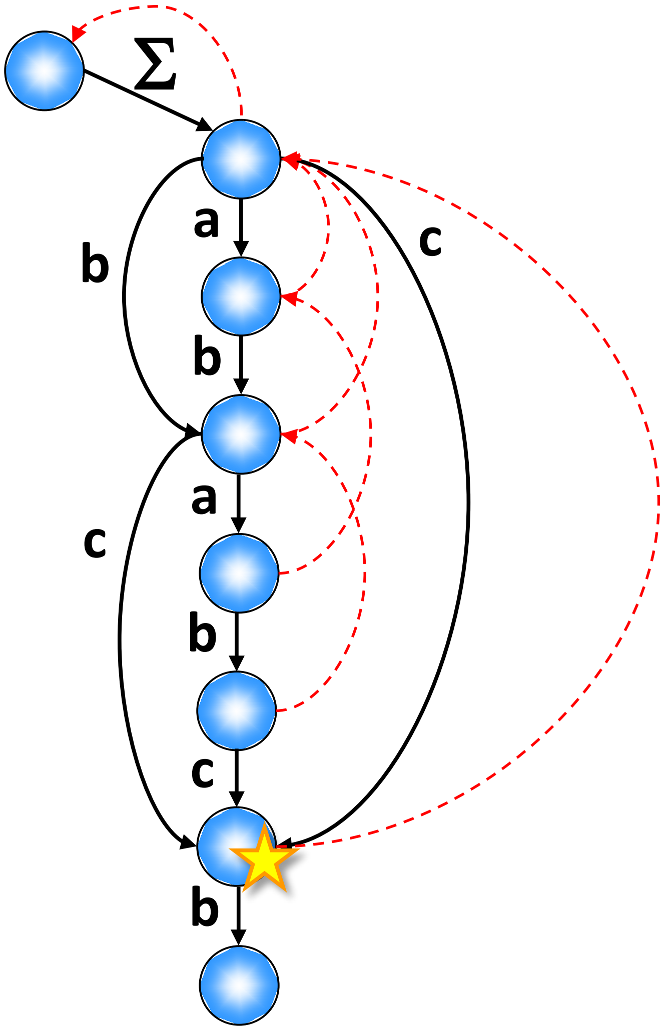

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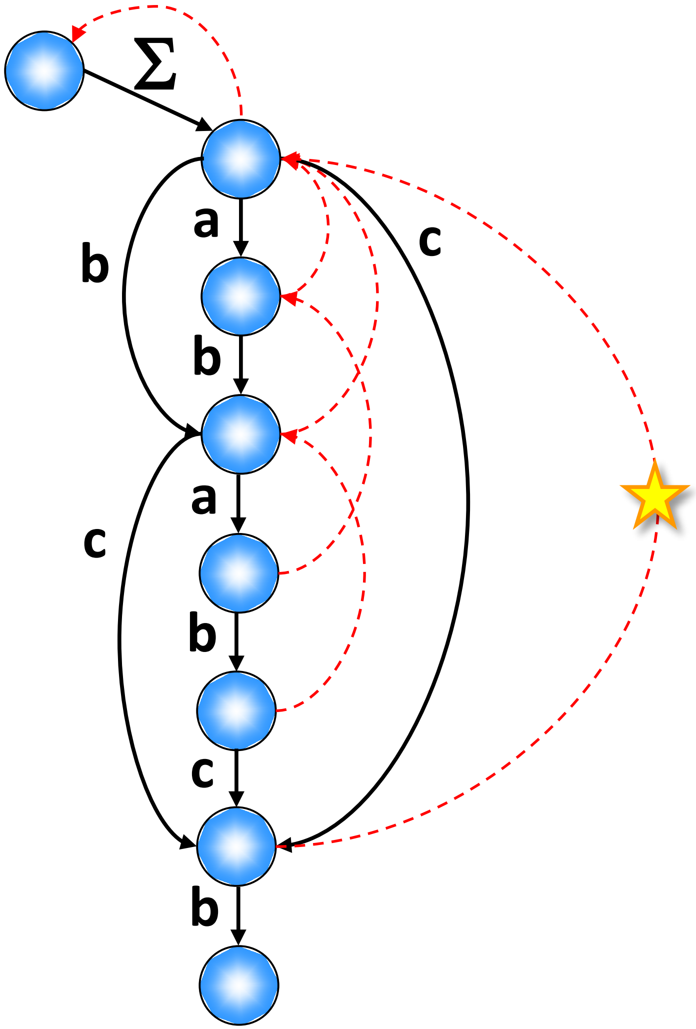

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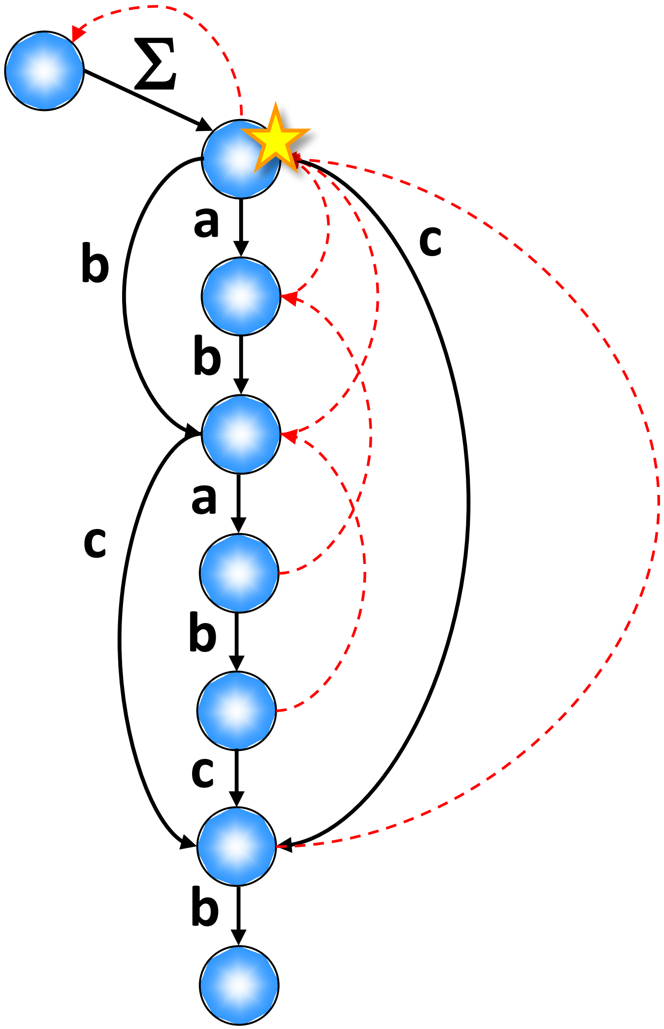

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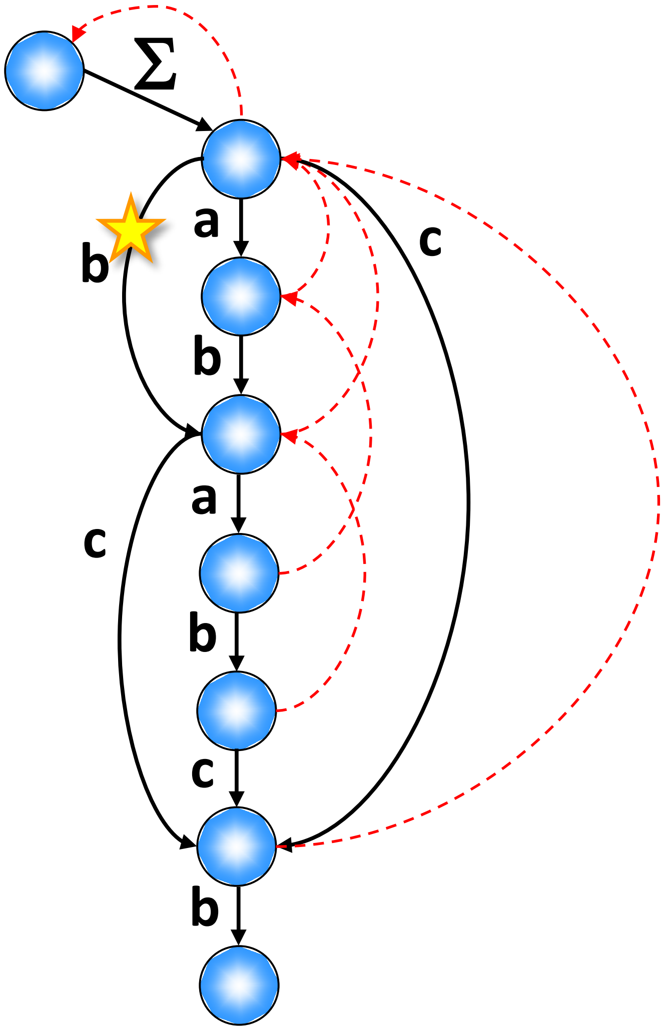

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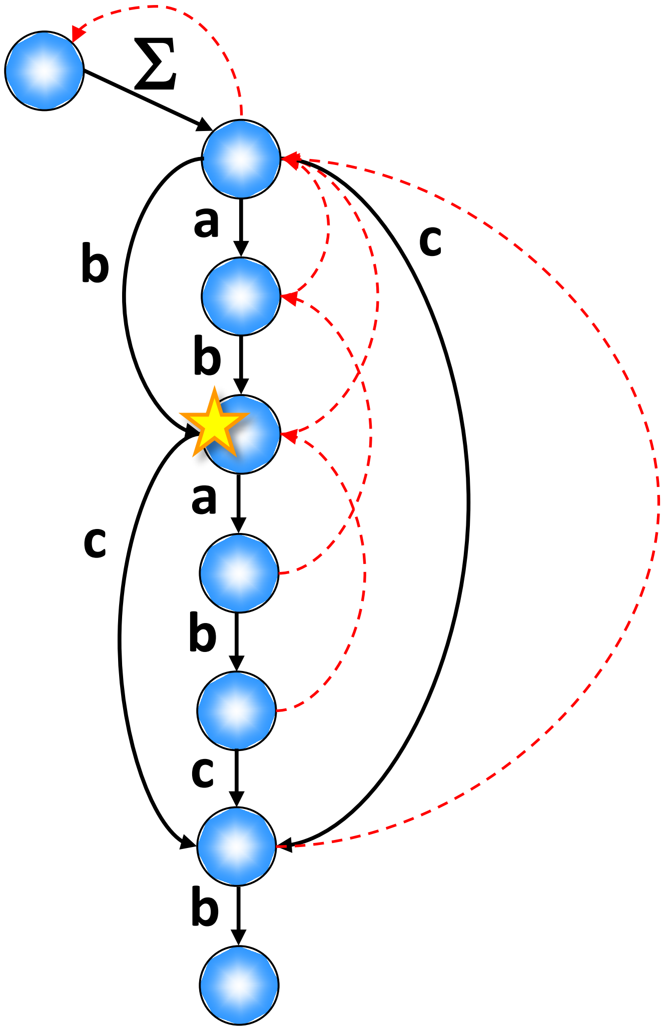

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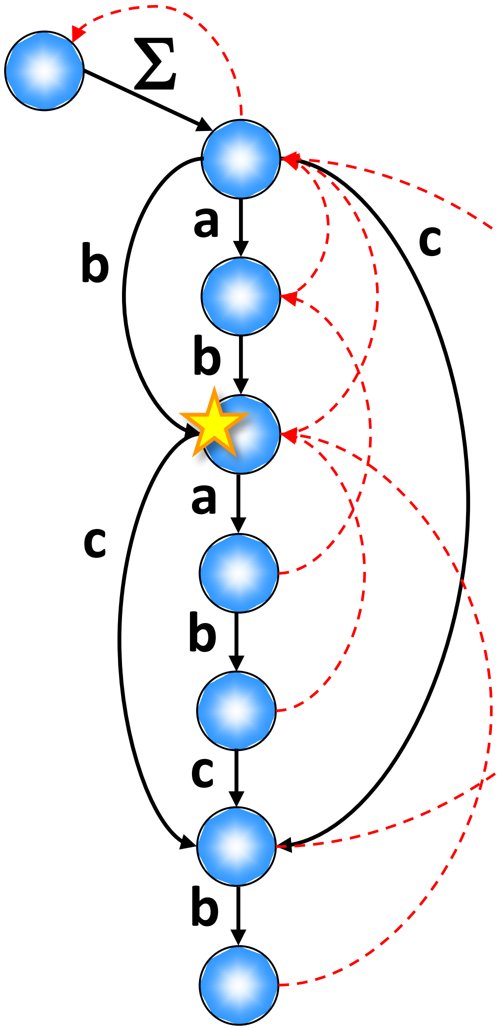

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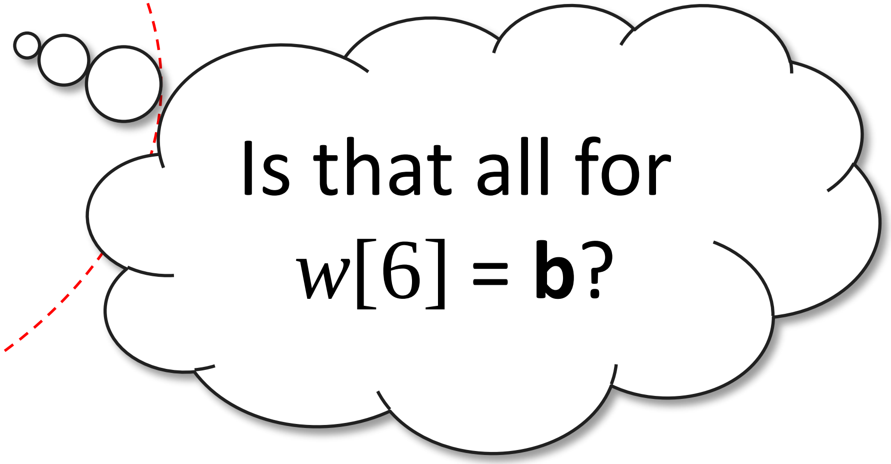
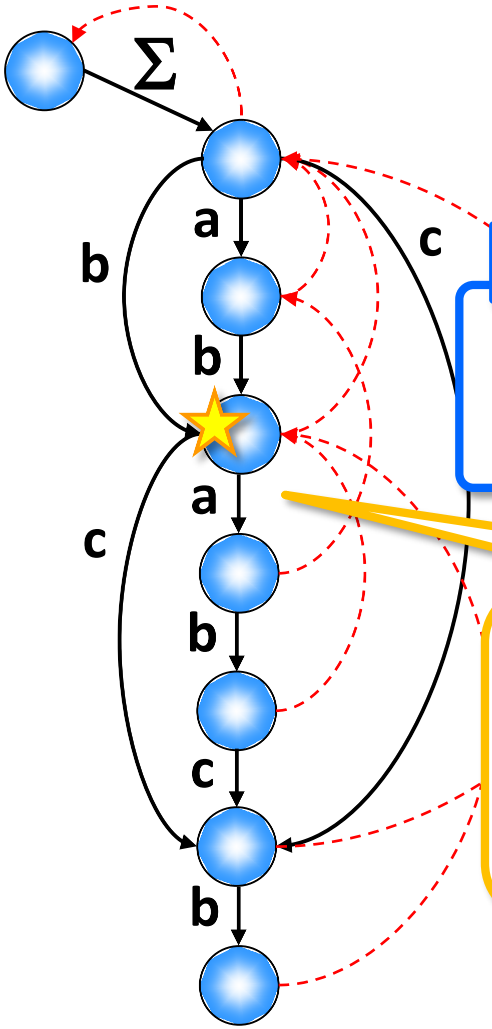

Is that all for $w[6] = \mathbf{b}$?

Online Construction of DAWGs



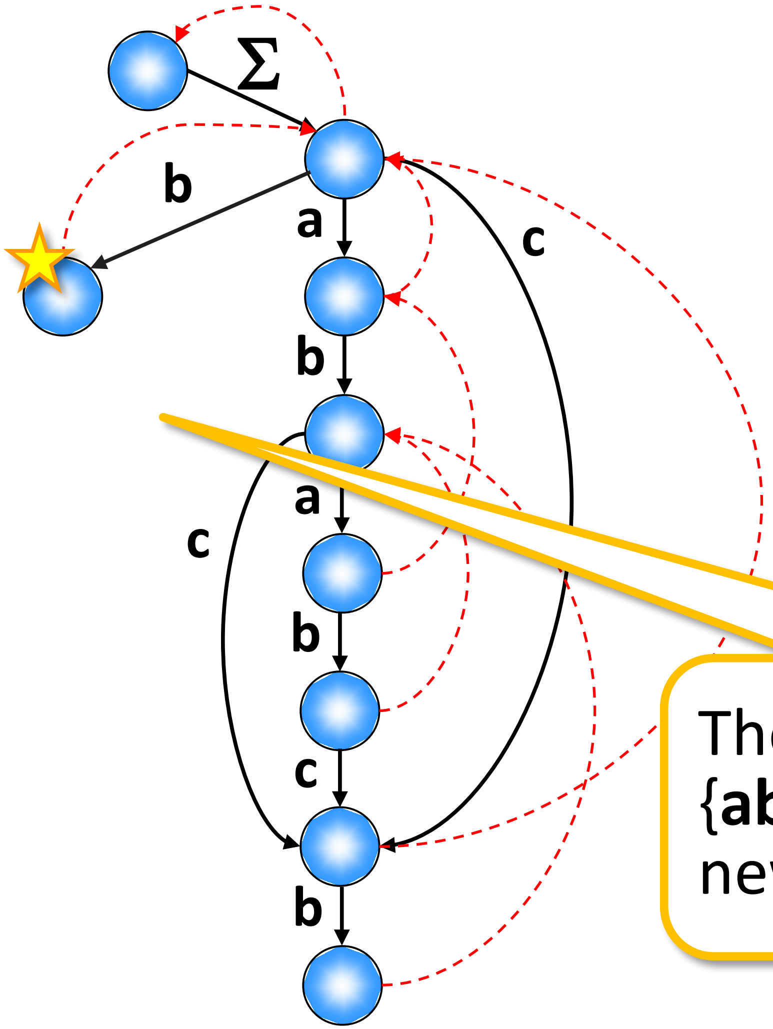
ababcb

Theorem 1

The DAWG of string w is the smallest automaton that **recognizes all suffixes** of w.

b is a suffix of $w[1..6]$ = **ababcb**, but **ab** isn't.
So, node {**ab**, **b**} must be split.

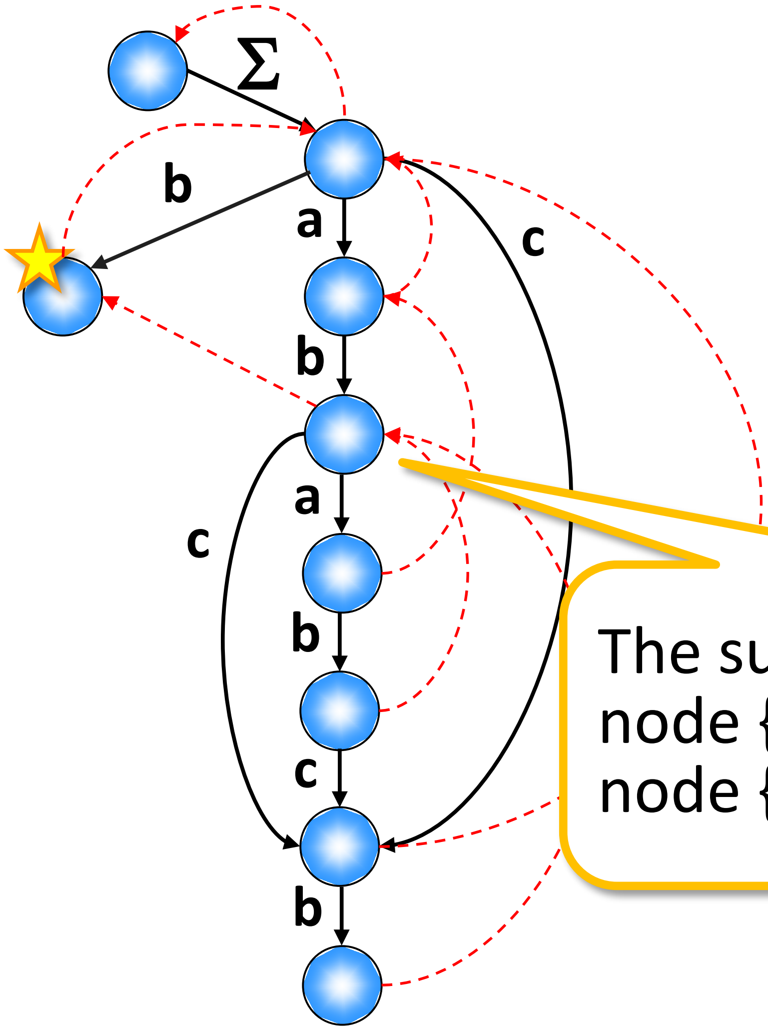
Online Construction of DAWGs



ababcb

The suffix link of old node {**ab**, **b**} is inherited by the new node {**b**}.

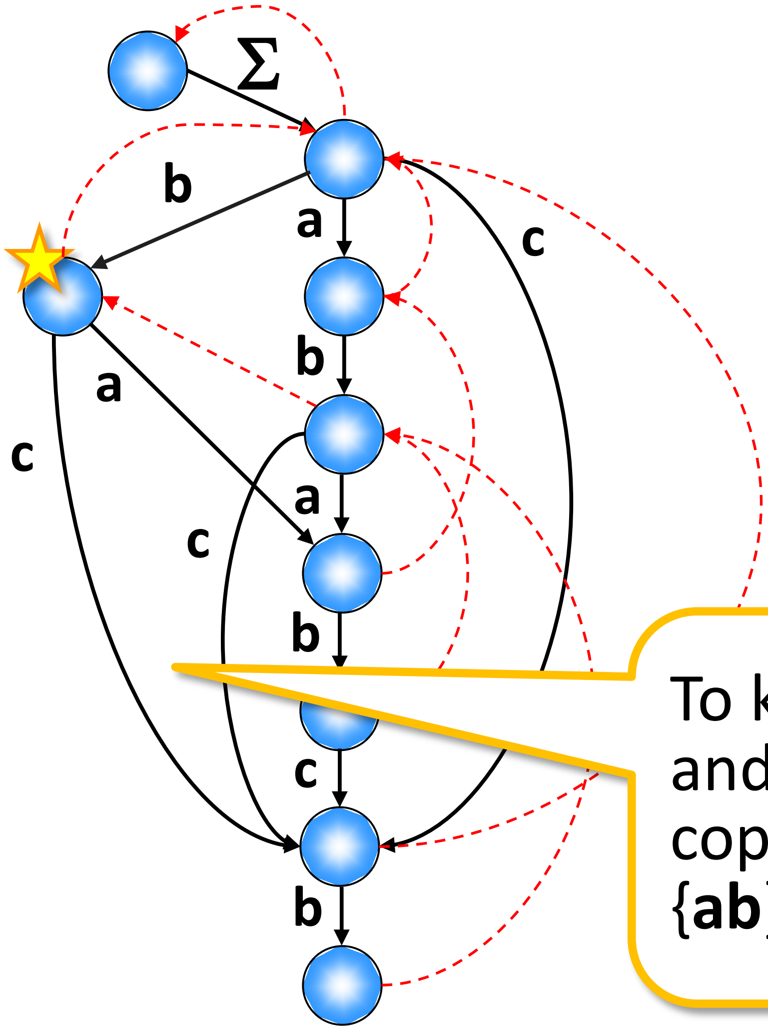
Online Construction of DAWGs



ababcb

The suffix link from the other node {**ab**} points to the new node {**b**}.

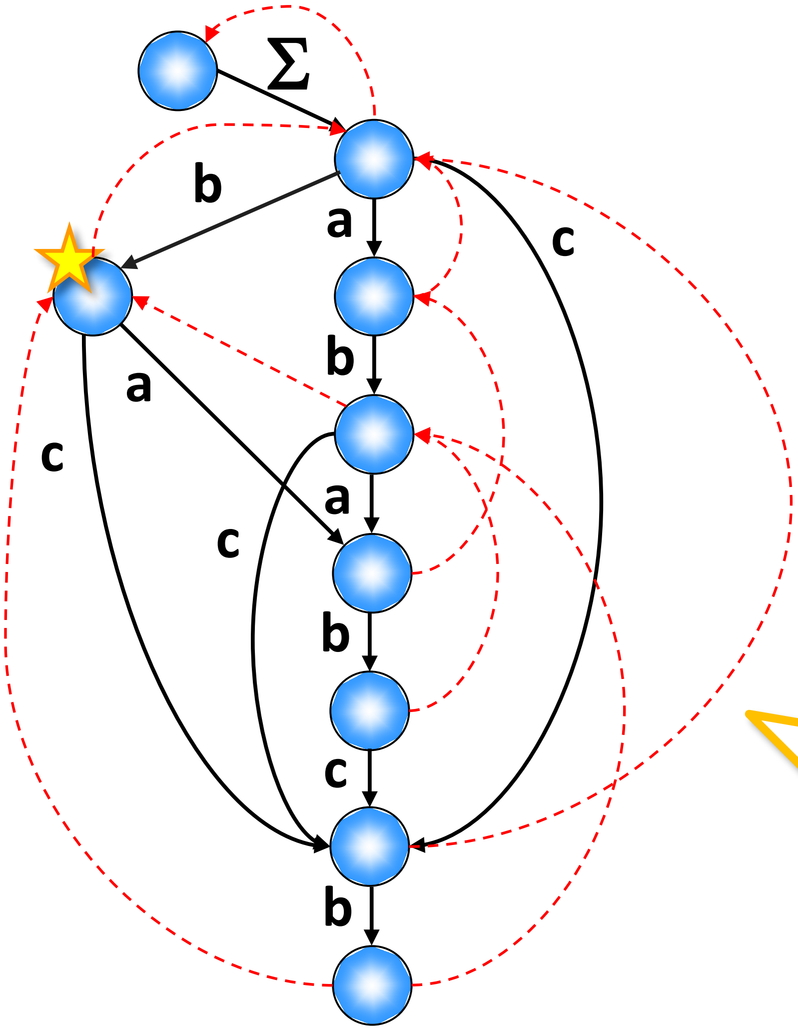
Online Construction of DAWGs



ababcb

To keep the path for **babcb** and **bcb** from node {**b**}, we copy the out-edges of node {**ab**}.

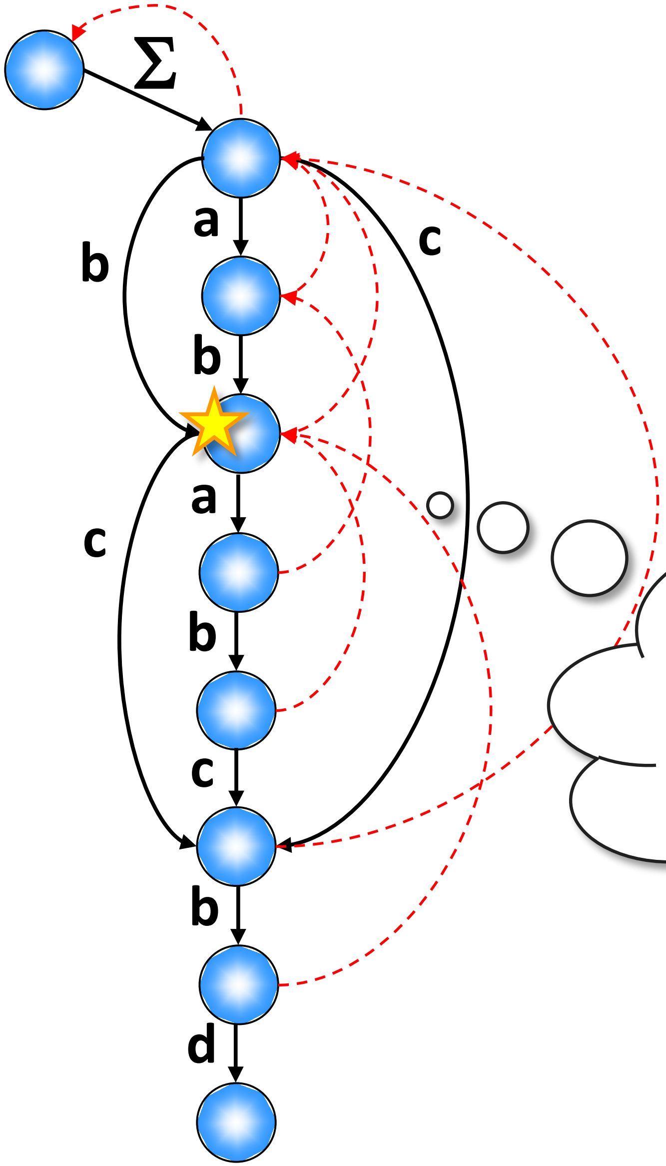
Online Construction of DAWGs



ababcb

The DAWG for string **ababcb** is complete.

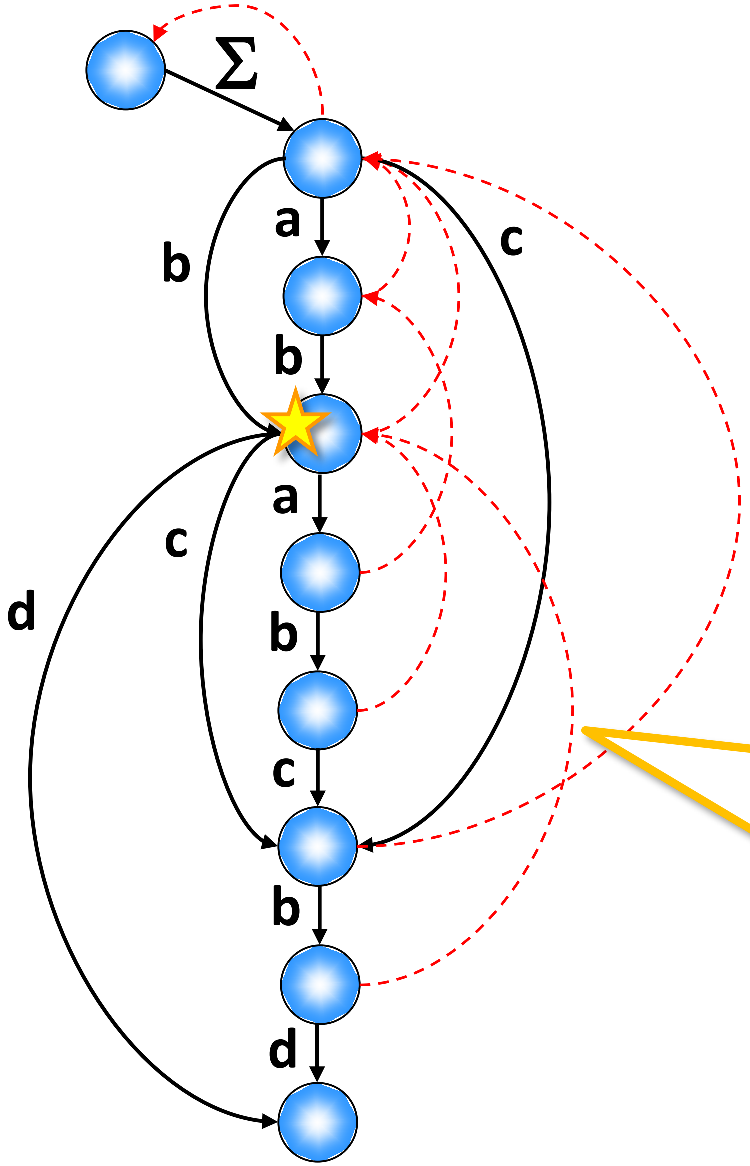
Node Split is Essential



ababcbd

What would happen if we did not split the node {**ab**, **b**} ?

Node Split is Essential



ababcbd

Now the DAG has a path for **abd**, but this does not appear in string **ababcbd**!

Online Construction of DAWGs

Theorem 6

The DAWG of a given string of length n can be constructed online (left-to-right) in $O(n \log \sigma)$ time, where $\sigma = |\Sigma|$.

- Clearly, the amount of work is proportional to the numbers of nodes, edges, and suffix links in the DAWG, each of which is $O(n)$.
- The $\log \sigma$ factor is to maintain BSTs for searching branches.

Left-to-right Construction of DAWG ⇒ Right-to-left Construction of Suffix Tree

Corollary 3

The suffix tree of a given string of length n can be constructed online (right-to-left) in $O(n \log \sigma)$ time.

- Immediate from Theorem 6.
- This corollary generalizes Weiner's right-to-left suffix tree construction.

DAWG Construction for Integer Alphabets

Theorem 7

The edge-sorted DAWG of a given string w of length n over an integer alphabet $\Sigma = \{1, \ldots, n^{O(1)}\}$ can be constructed in $O(n)$ time.

- Build the suffix tree of w with suffix links in $O(n)$ time [Farach-Colton et al., 2000].
- Build DAWG with suffix links from suffix tree.
- Edges can be sorted in $O(n)$ time by bucket sort.

Recommended Reading (1/3)

- "The Smallest Automaton Recognizing the Subwords of a Text", Blumer et al., *TCS*, 1985.
  - Introduced DAWGs.
  - Duality with suffix trees.
  - Online $O(n \log \sigma)$-time DAWG construction algorithm.
- *Text Algorithms*, Crochemore & Rytter, Oxford University Press, 1994.
  - Text book. Chapter 6 is devoted for DAWGs.
  - Free(!) copy is available online at http://www.mimuw.edu.pl/~rytter/BOOKS/text-algorithms.pdf

Recommended Reading (2/3)

- "Automata and Forbidden Words", Crochemore et al., *IPL*, 1998.
  - DAWG-based $O(\sigma n)$-time algorithm for finding all MAWs.
- "Linear-Time Sequence Comparison Using Minimal Absent Words & Applications", Crochemore et al., LATIN, 2016.
  - String similarity measure based on MAWs.

Recommended Reading (3/3)

- “Computing DAWGs and Minimal Absent Words in Linear Time for Integer Alphabets”, Fujishige et al., *MFCS*, 2016 (to appear).
  - Offline $O(n)$-time DAWG construction algorithm for integer alphabets of size $n^{O(1)}$.
  - $O(n + |MAW_w|)$-time algorithm for finding all MAWs.
- “Fully-online Construction of Suffix Trees for Multiple Texts”, Takagi et al., *CPM*, 2016.
  - Fully-online $O(n \log \sigma)$ -time DAWG construction algorithm for multiple strings.